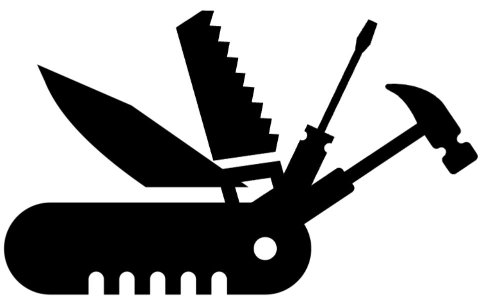
A Multi-tool for your Quantum Algorithmic Toolbox



Shelby Kimmel

Middlebury College

Based on work with

- **Chloe Ye, Da-Yeon Koh, Noel Anderson, Jay-U Chung** arXiv:2012.01276
- **Teal Witter** arXiv:2010.02324 (WADS 2021)
- **Kai DeLorenzo, Teal Witter**, arXiv:1904.05995 (TQC 2019)
- Michael Jarret, Stacey Jeffery, Alvaro Piedrafita, arXiv:1804.10591 *(*ESA 2018*)*
- Stacey Jeffery: arXiv: 1704.00765 (Quantum vol 1 p 26)
- **Bohua Zhan**, Avinatan Hassidim, arXiv:1101.0796 (ITCS 2012)

Quantum Computers

Quantum computers use quantum particles – atoms, photons, phonons, electrons, etc. – to compute.

Famously fast for:

- factoring
- simulating quantum chemistry

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- **Specialized Problems**
- **Sophisticated design/analysis**

Quantum Computers

Quantum computers use quantum particles – atoms, photons, phonons, electrons, etc. – to compute.

Famously fast for:

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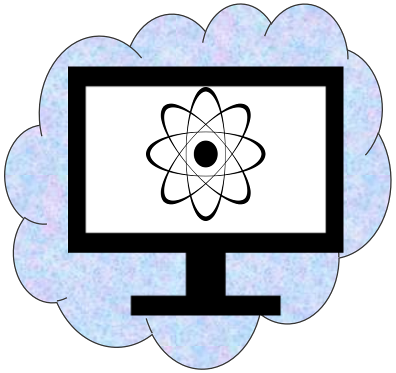
- **Specialized Problems**
- **Sophisticated design/analysis**

- **Everyday Problems?**
- **Accessible design/analysis?**

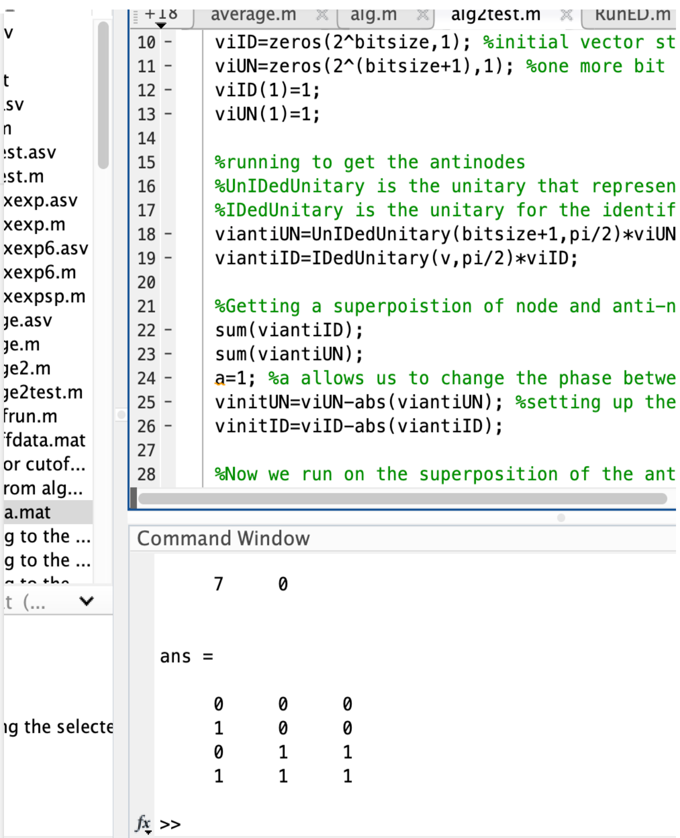
Imagine…

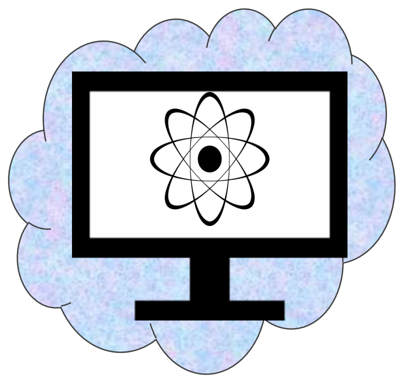
Year 2027



Imagine...

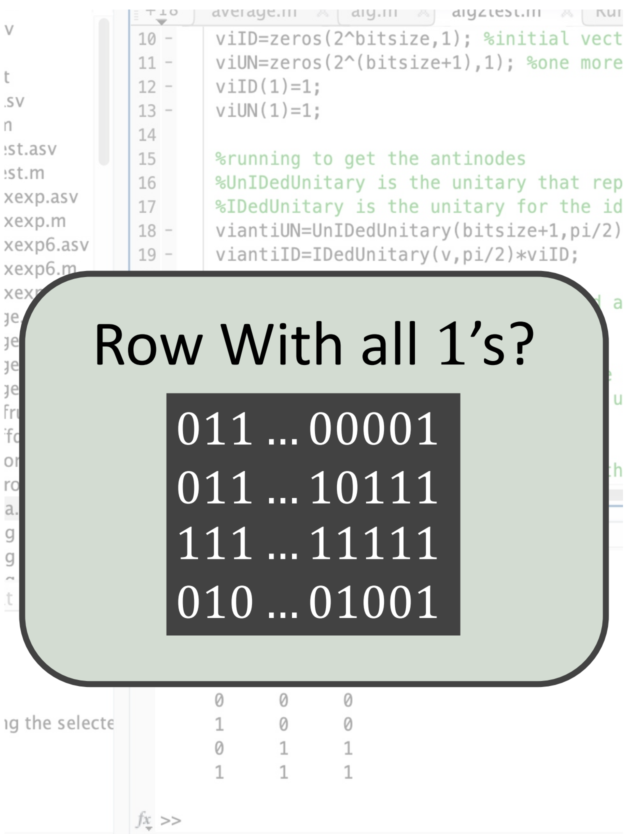
Year 2027

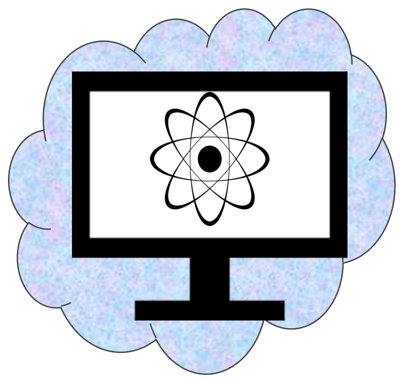




Imagine...

Year 2027





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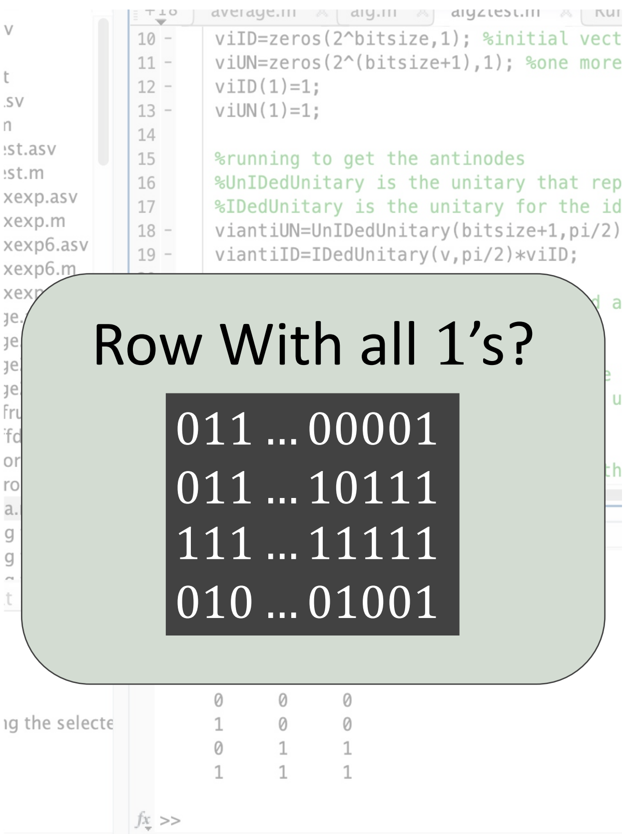
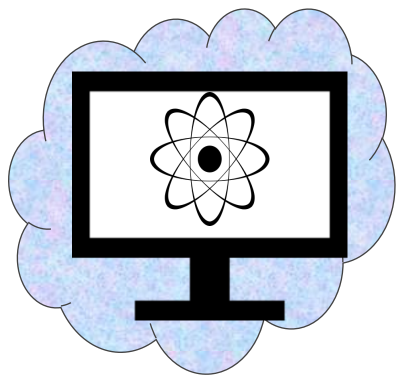
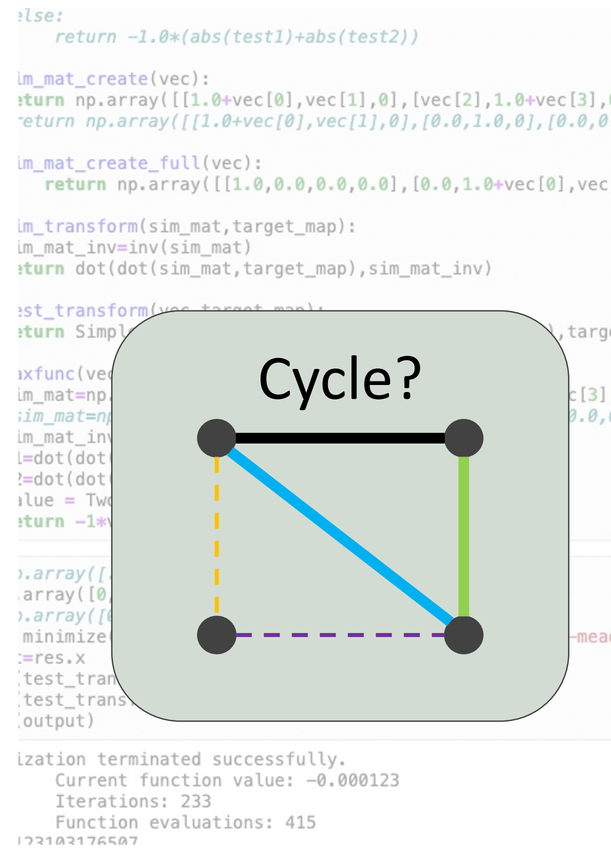
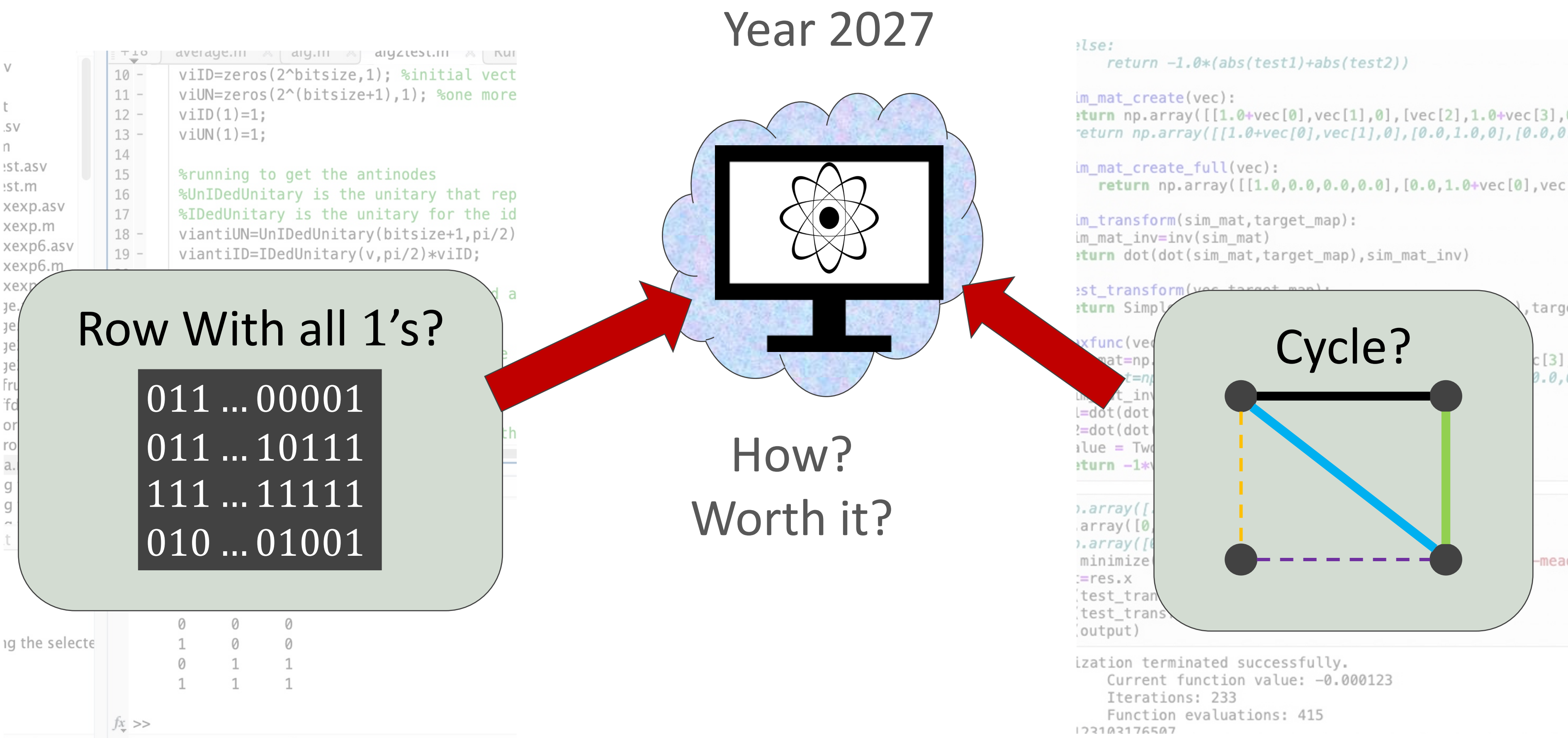


Year 2027

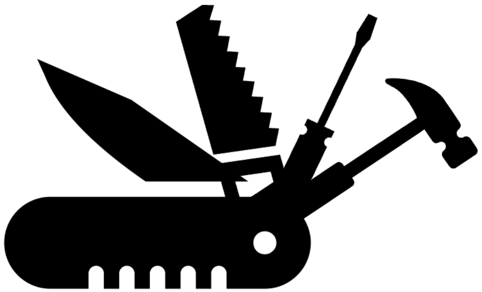


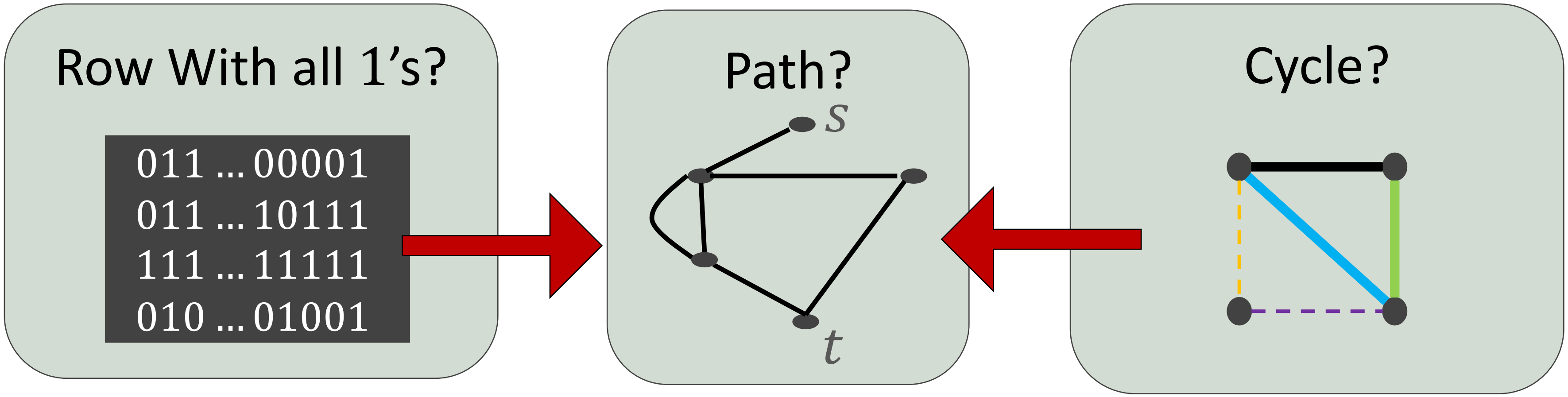


Imagine...


Year 2027
Row With all 1's?
011 … 00001
011 … 10111
111 … 11111
010 … 01001
Cycle?
How?
Worth it?

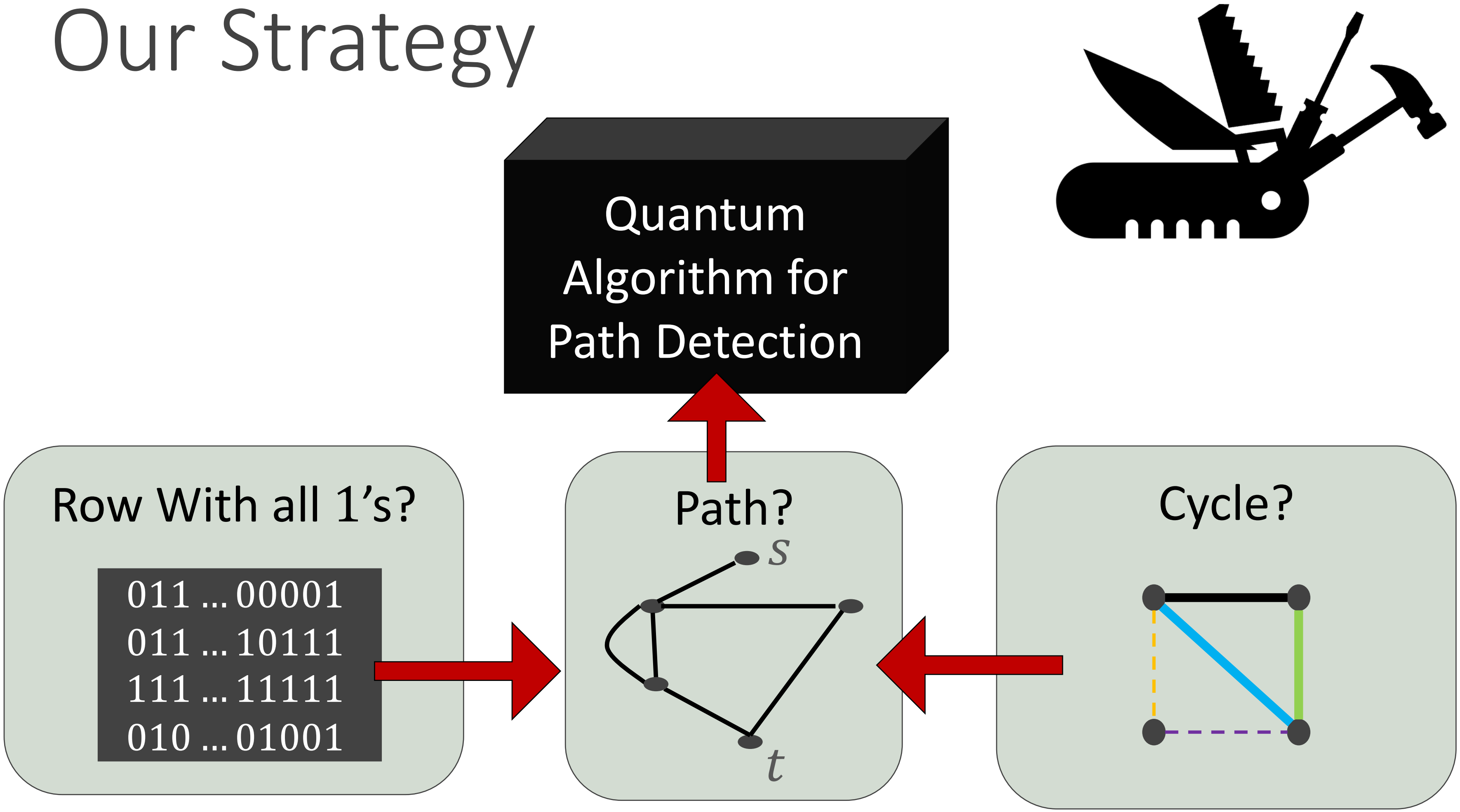
Our Strategy





Reduce our problem to path detection

Our Strategy

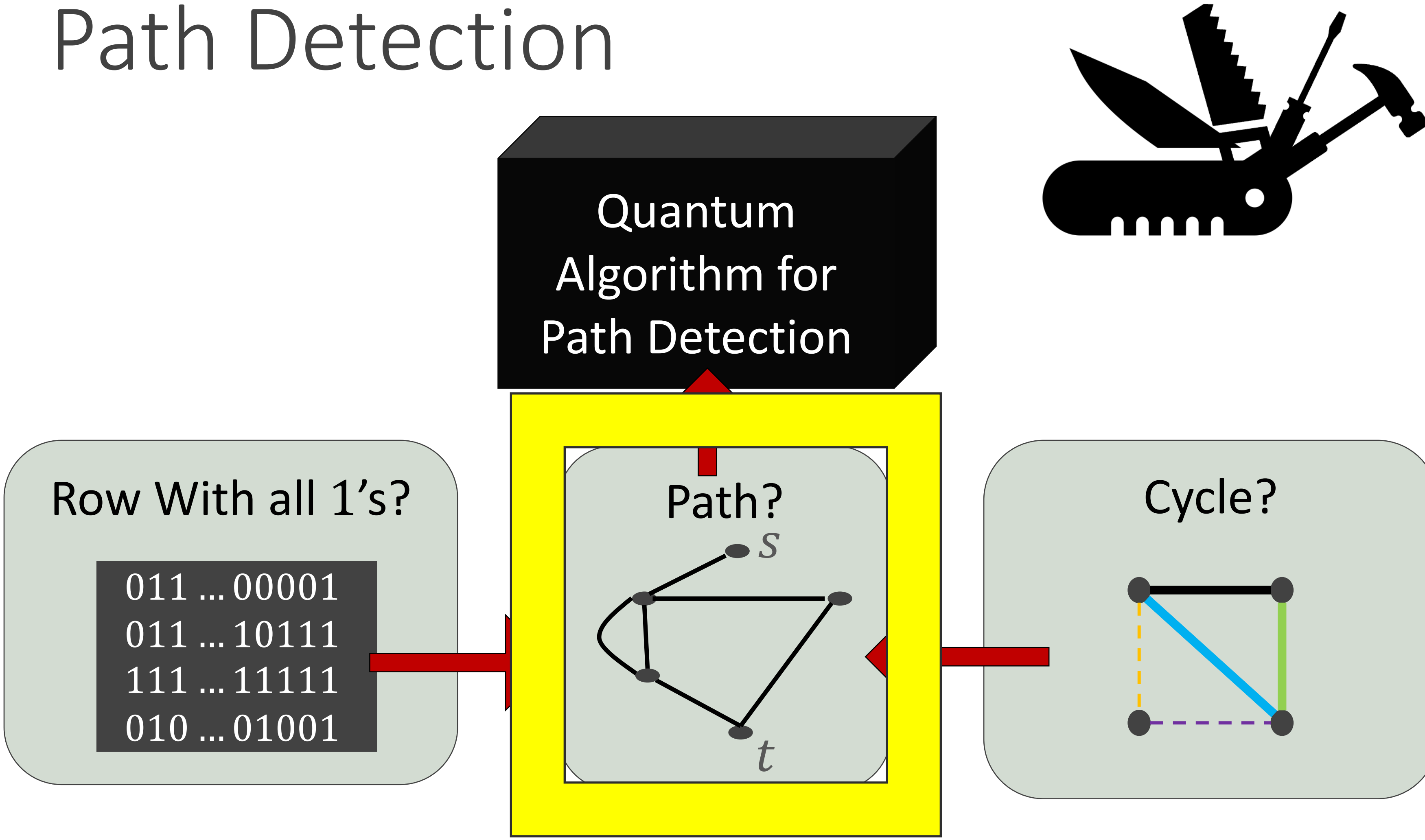


Reduce our problem to path detection

Outline

1. Path Detection (set-up)
2. Reductions
3. Quantum Path Detection Algorithm (complexity)

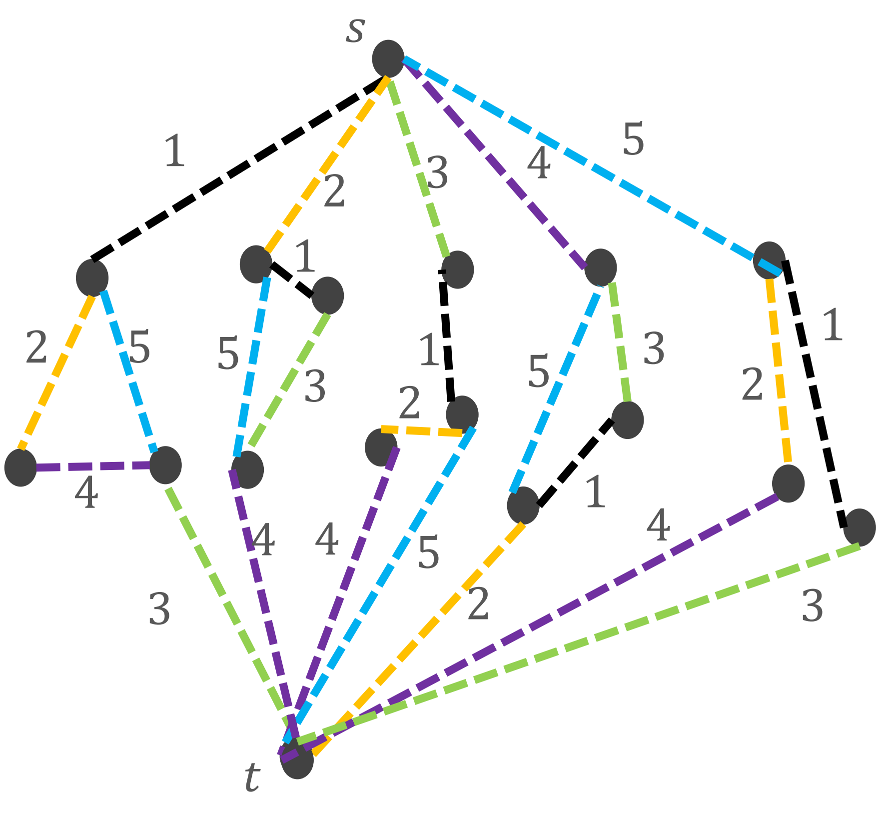
Path Detection



Reduce our problem to path detection

Path Detection

Is there a path from s to t?



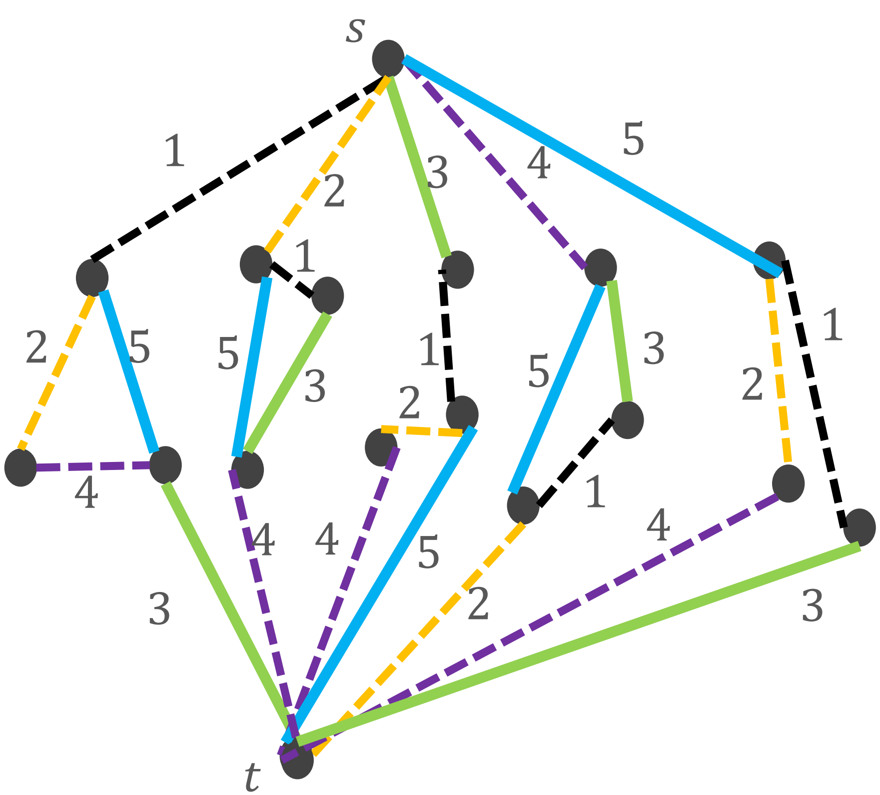
“Skeleton graph” G

Bit String Input x:

x_1	x_2	x_3	x_4	x_5

Path Detection

Is there a path from s to t?



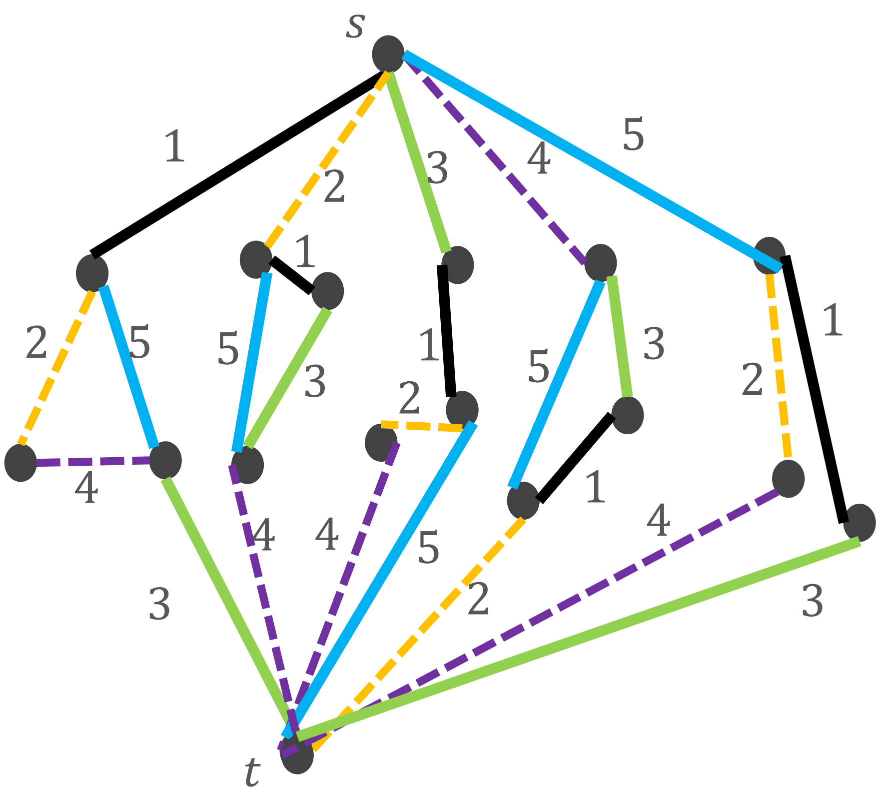
$G(x)$

Bit String Input x:

x_1	x_2	x_3	x_4	x_5
0	0	1	0	1

Path Detection

Is there a path from s to t?



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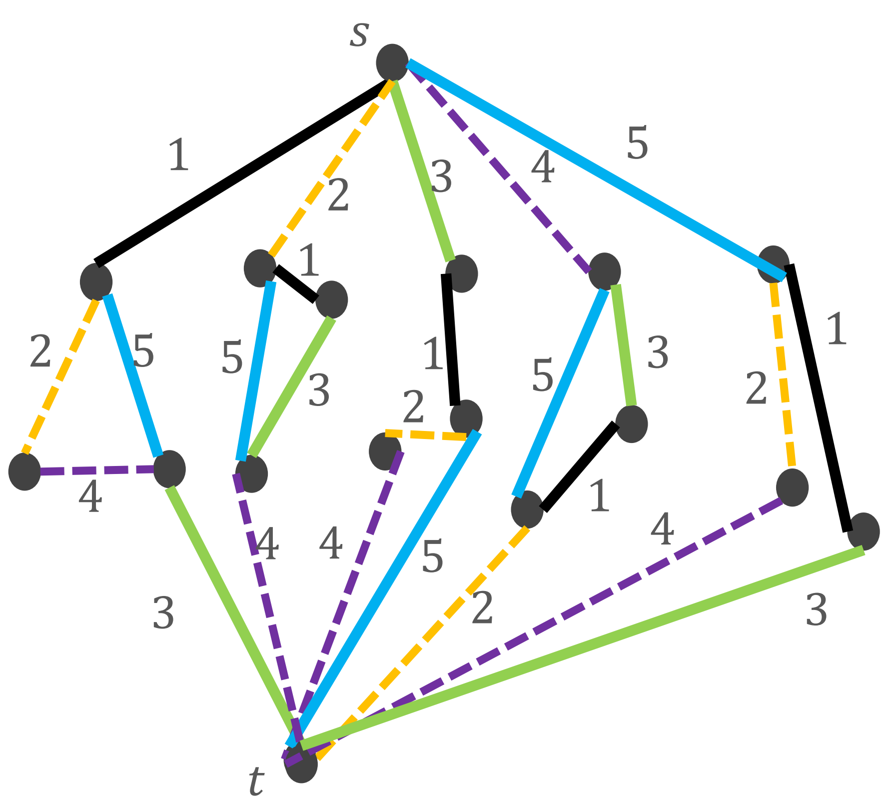


$G(x)$

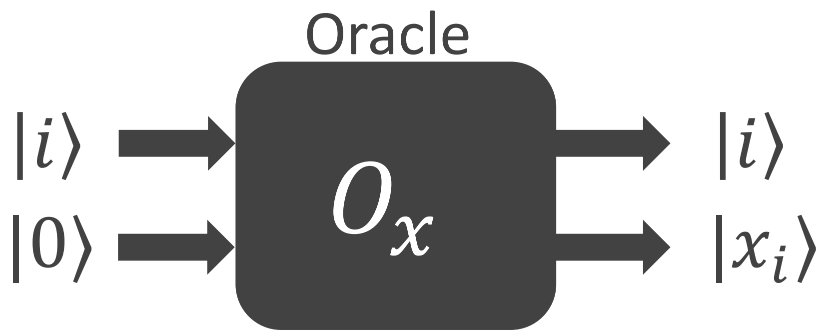
Bit String Input x:

x_1	x_2	x_3	x_4	x_5
1	0	1	0	1

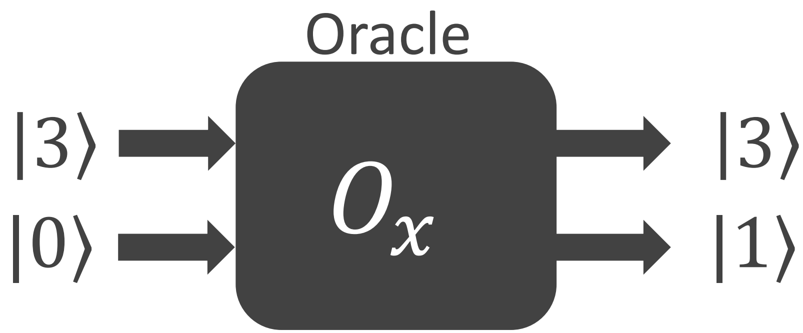
Catch:

- Bit string initially hidden

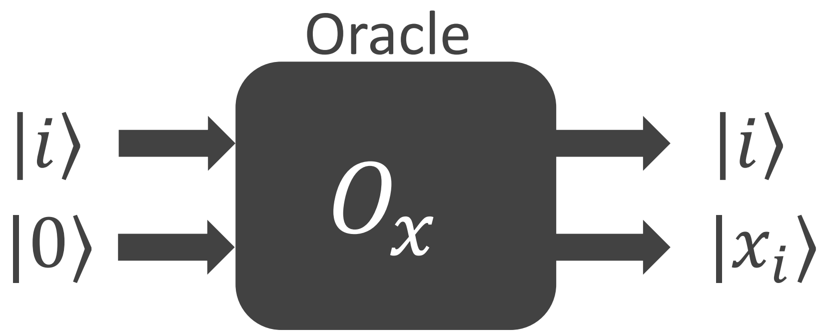
Quantum Query



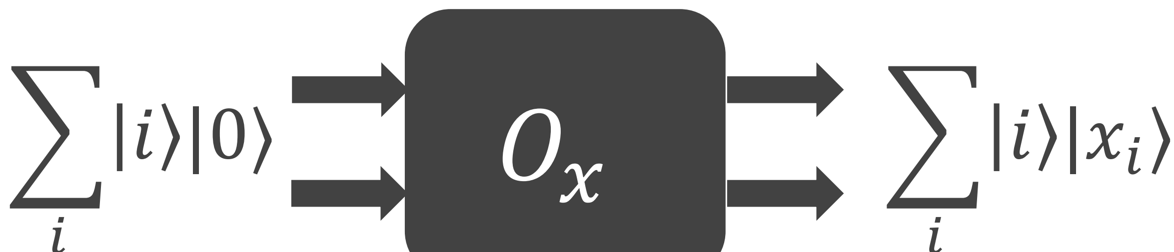
Quantum Query



Quantum Query

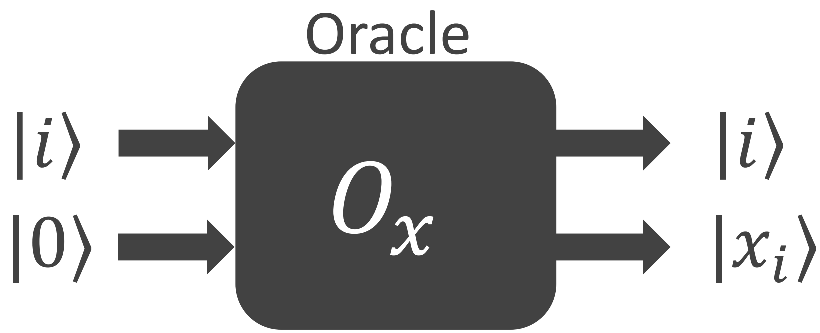


Query in superposition



(states not normalized)

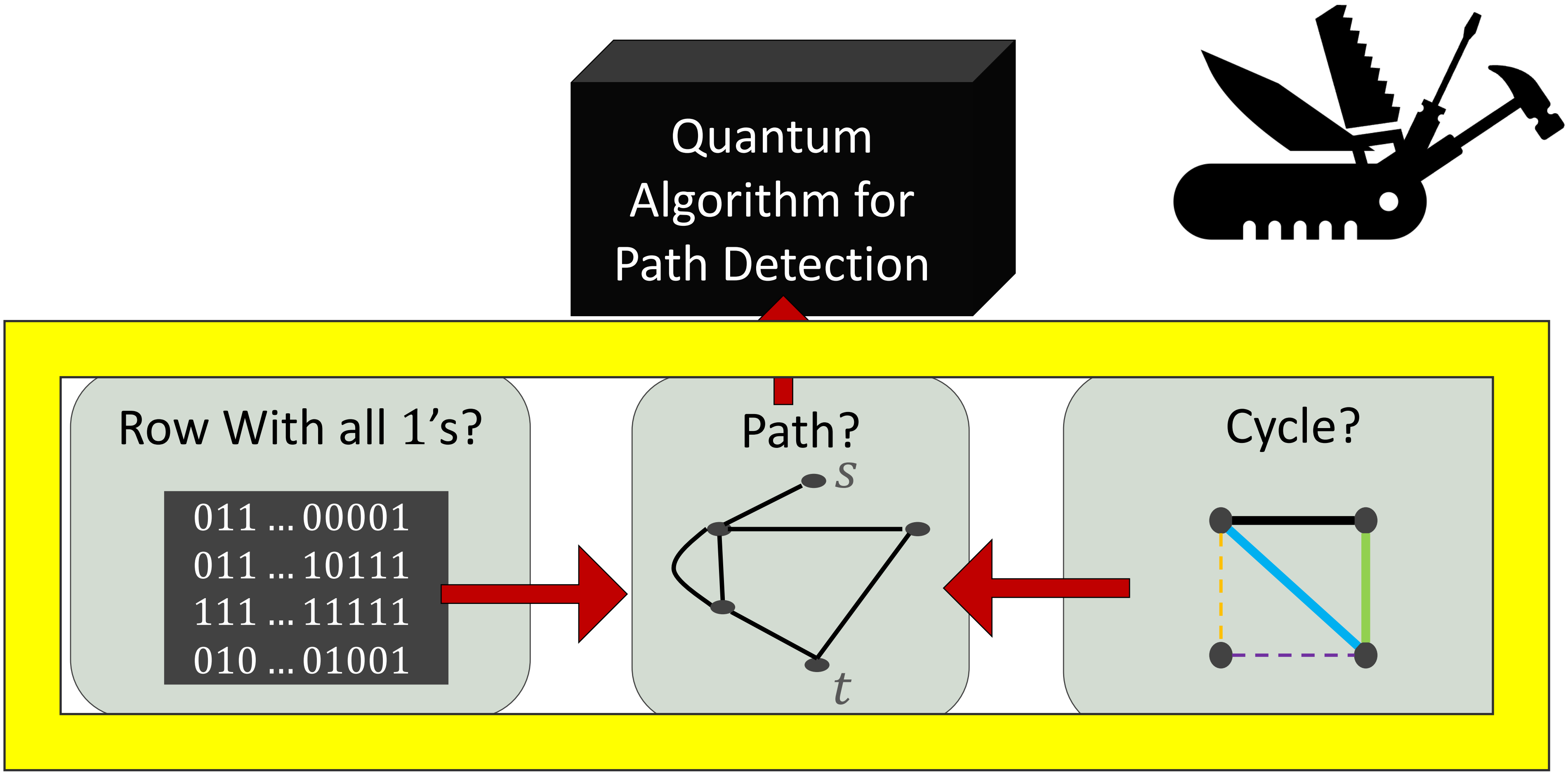
Quantum Query



Goal: Figure out if path while using the oracle as few times as possible (while revealing as few bits as possible)
-> minimize **Query Complexity**

For our algorithms, runtime scales like query complexity, times the time it takes to do one step of a quantum walk on the skeleton graph (for most graphs with structure: log time)

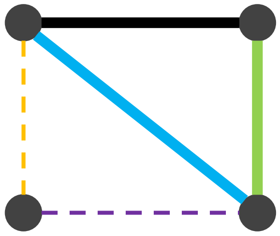
Reduction to Path Detection



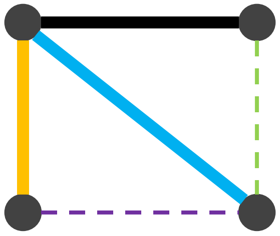
Reduce our problem to path detection

Cycle Detection to Path Detection

Problem: Is there a cycle?



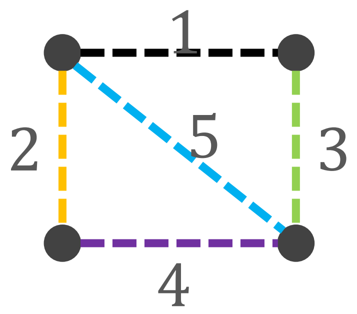
Yes



No

Cycle Detection to Path Detection

Problem: Is there a cycle?



Unknown bit string

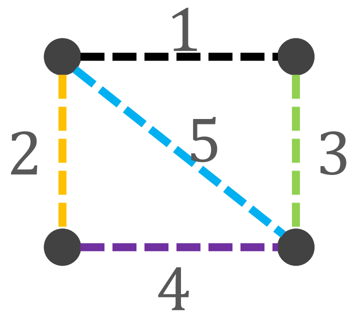
$x_1 x_2 x_3 x_4 x_5$

$x_i = 1 \leftrightarrow$ edge i is present

Cycle Detection to Path Detection

Problem: Is there a cycle?

Subproblem: Is there a cycle through edge 1?



There is a cycle through Edge 1 iff

- Edge 1 is present
- Path between the endpoints of Edge 1 not using Edge 1

AND

Cycle Detection to Path Detection

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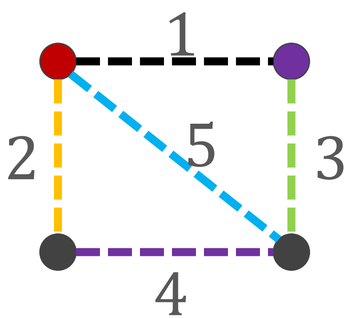
Subproblem: Is there a cycle through edge 1?



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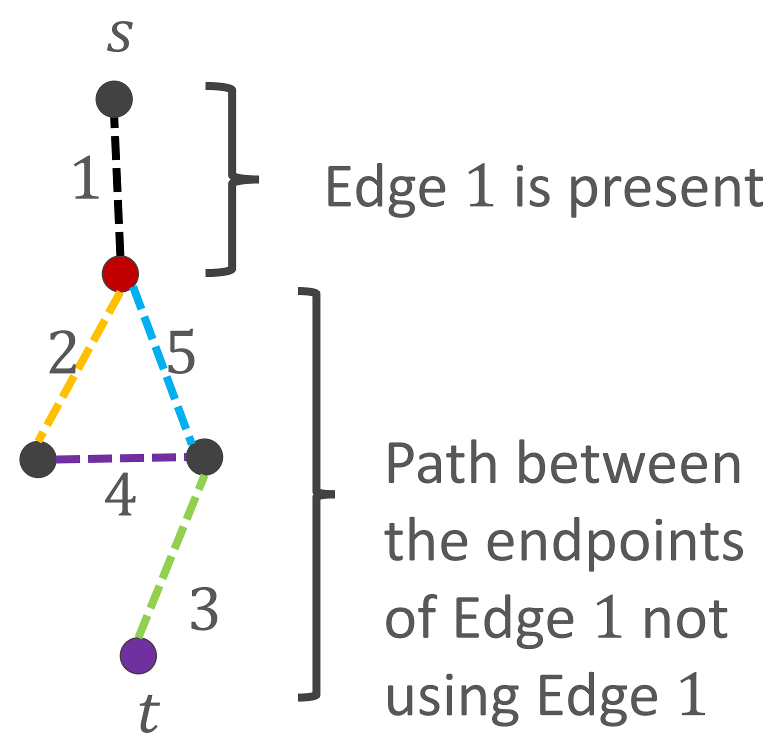
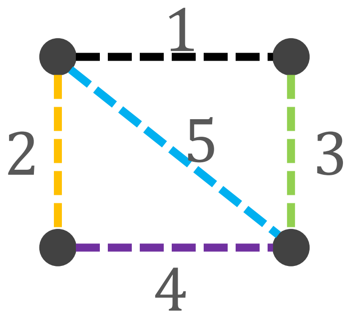


Cycle Detection to Path Detection

Problem: Is there a cycle?



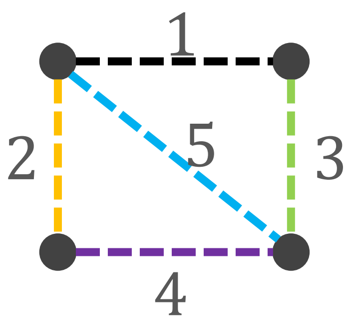
There is a cycle if

- Cycle through edge 1
- Cycle through edge 2
- …
- Cycle through edge n

Or

Cycle Detection to Path Detection

Problem: Is there a cycle?



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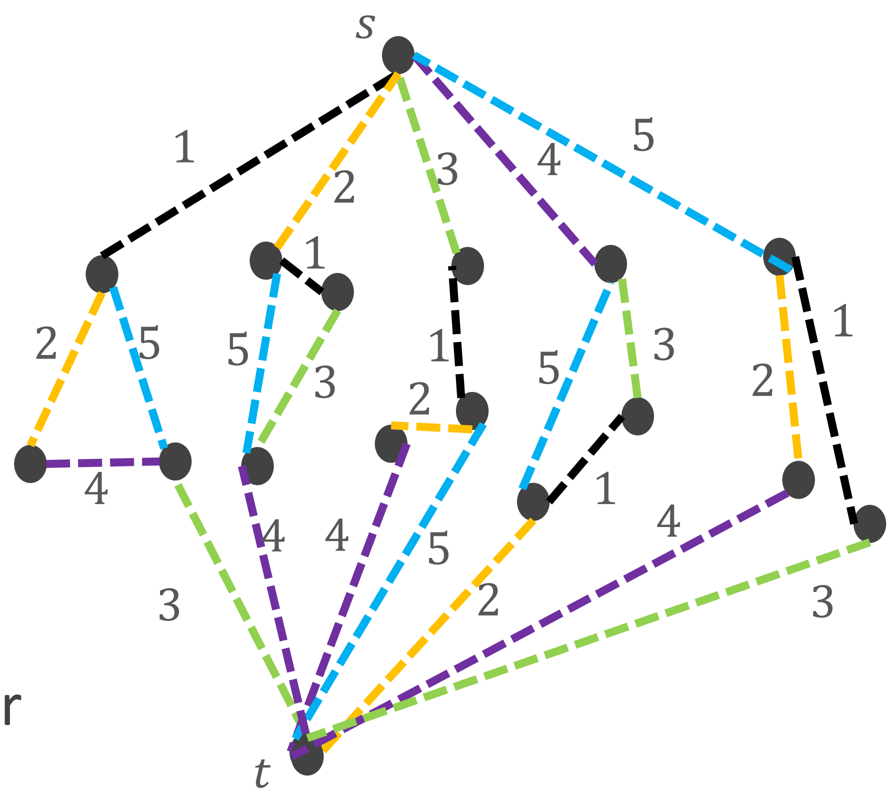
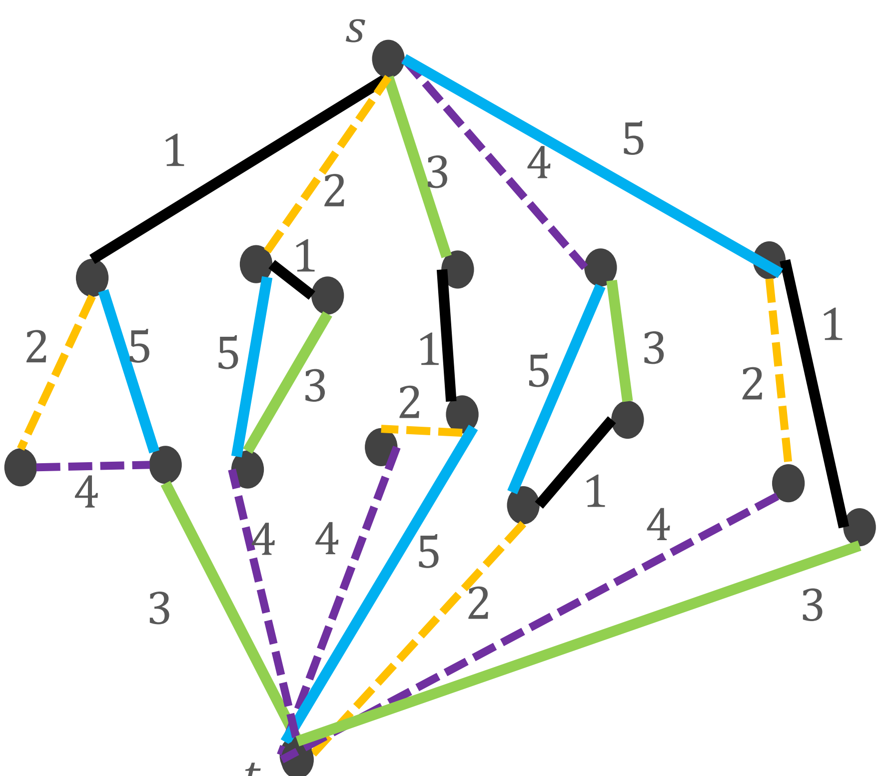


Cycle Detection to Path Detection

Is there a path from s to t?



$G(x)$

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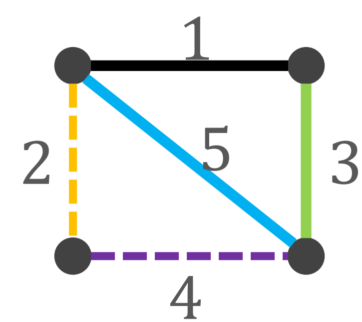
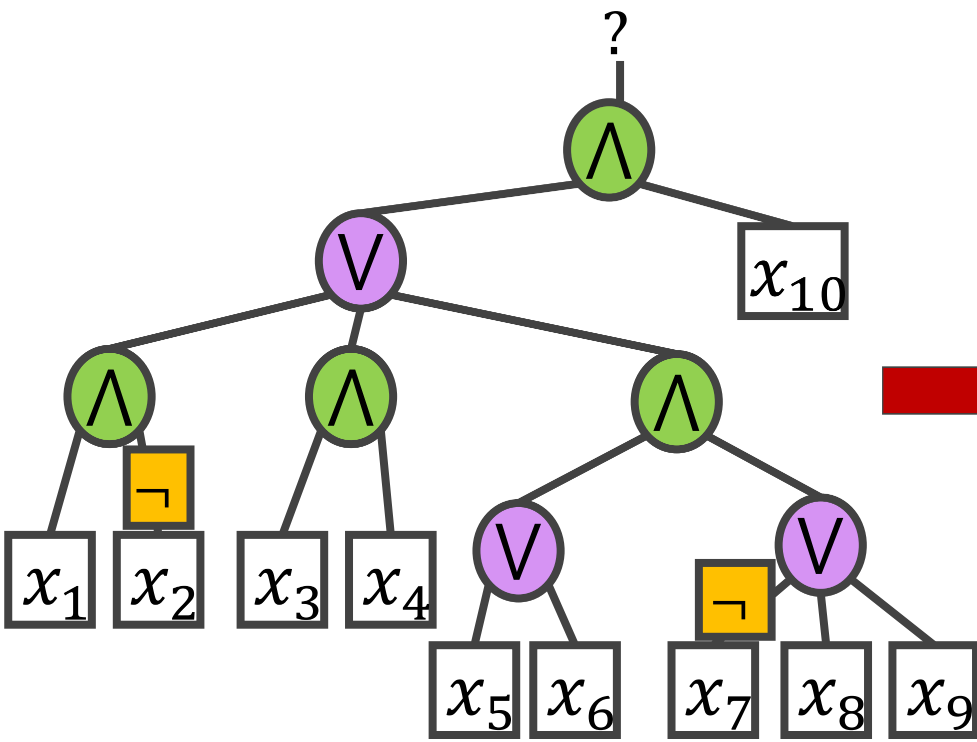
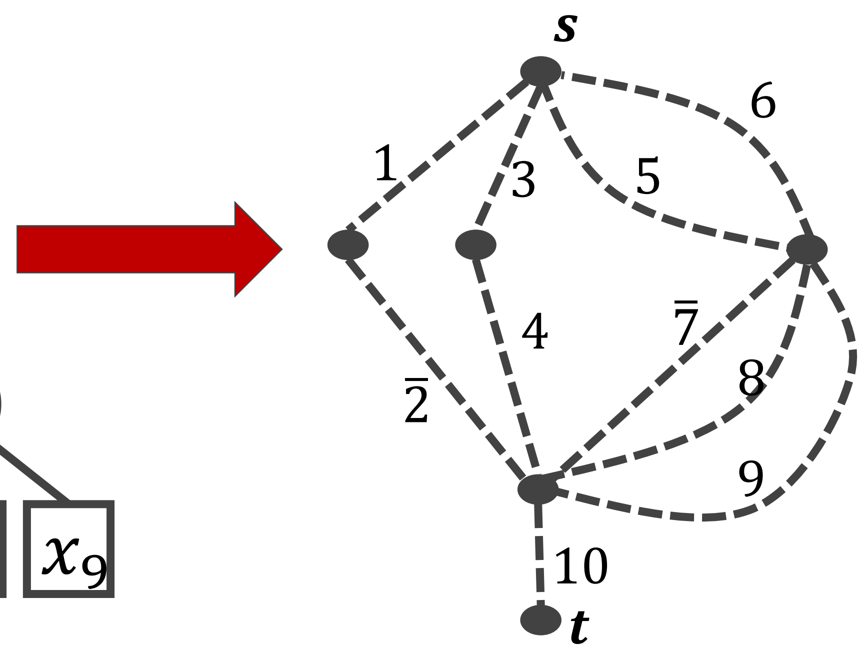


Formula Evaluation to Path Detection





[Nisan, Ta-Shma, ‘95]

Quantum Path Detection Reductions

- Read-once Boolean formulas [Jeffery, **K,** ‘17]
- Total connectivity [Jarret, Jeffery, **K,** Piedrafita, ‘18]
- Cycle detection [Delorenzo, **K,** Witter, ‘19]
- Even length cycle detection [Delorenzo, **K,** Witter, ‘19]
- Bipartiteness [Delorenzo, **K,** Witter, ‘19]
- Maximum matching (**K,** Witter, ‘21)
- 2-player game evaluation [Zhan, Hassidim, **K,** ‘12]
- Directed st-connectivity (Beigi, Taghavi `19)
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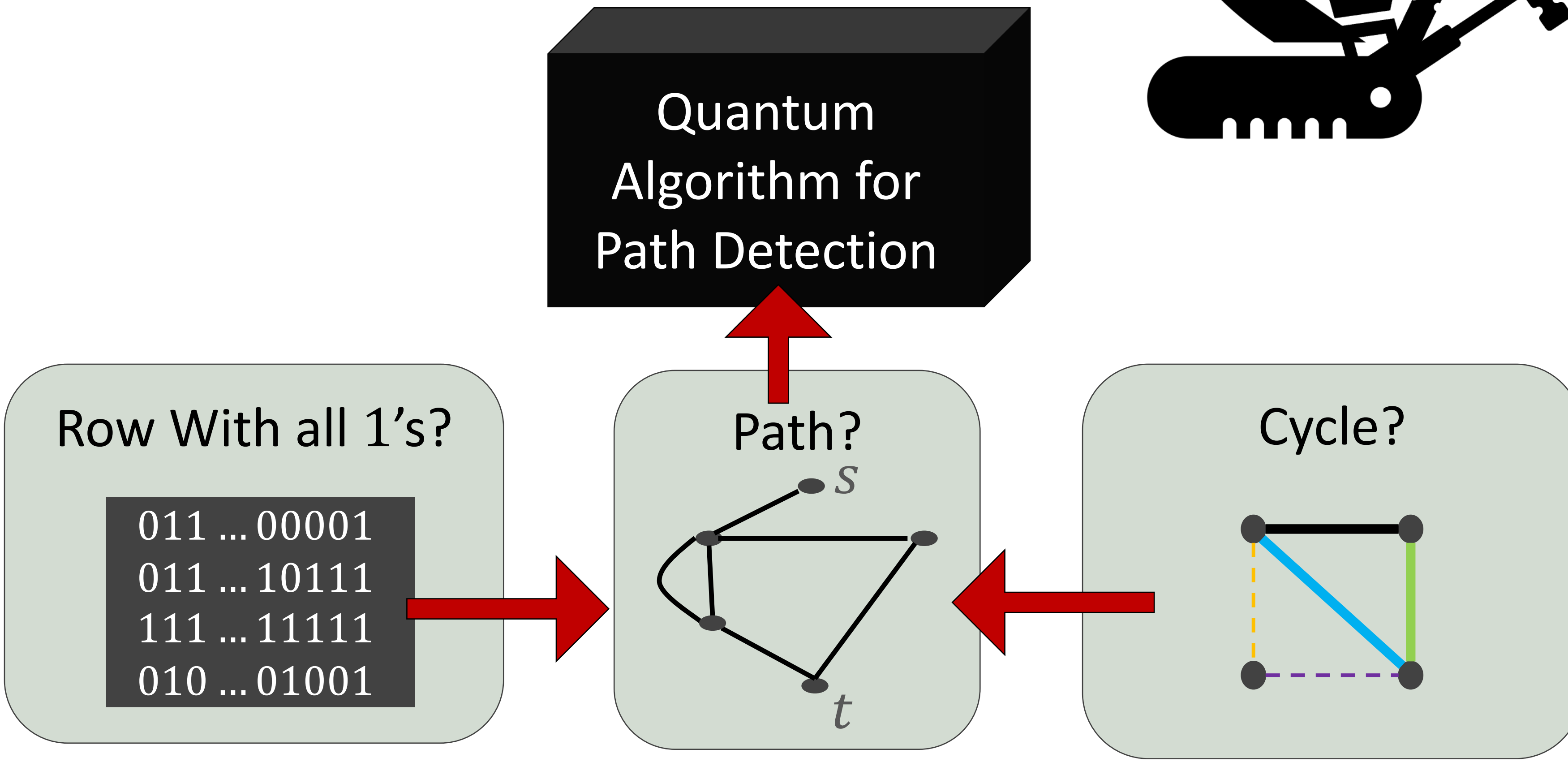
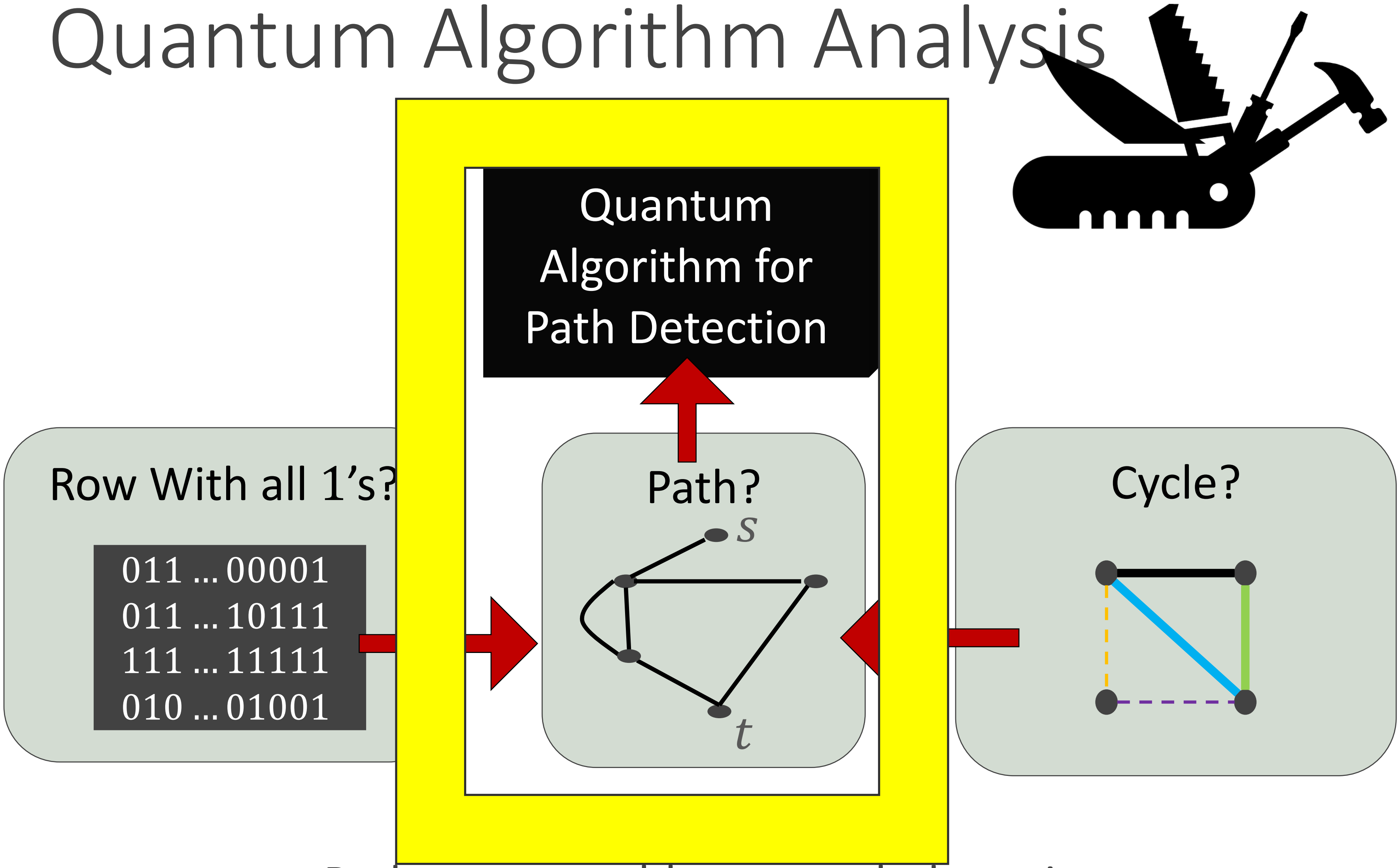
Quantum Algorithm Analysis



Reduce our problem to path detection

Quantum Algorithm Analysis



Reduce our problem to path detection

Quantum Path Detection Algorithm Complexity

Space Complexity: $O(\log(\#\ edges\ in\ skeleton\ graph))$

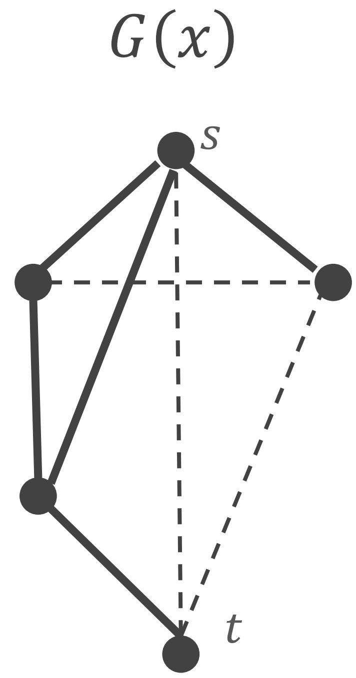
Quantum Path Detection Algorithm Complexity

Space Complexity: $O(\log(\#\ edges\ in\ skeleton\ graph))$

Query Complexity: Depends on

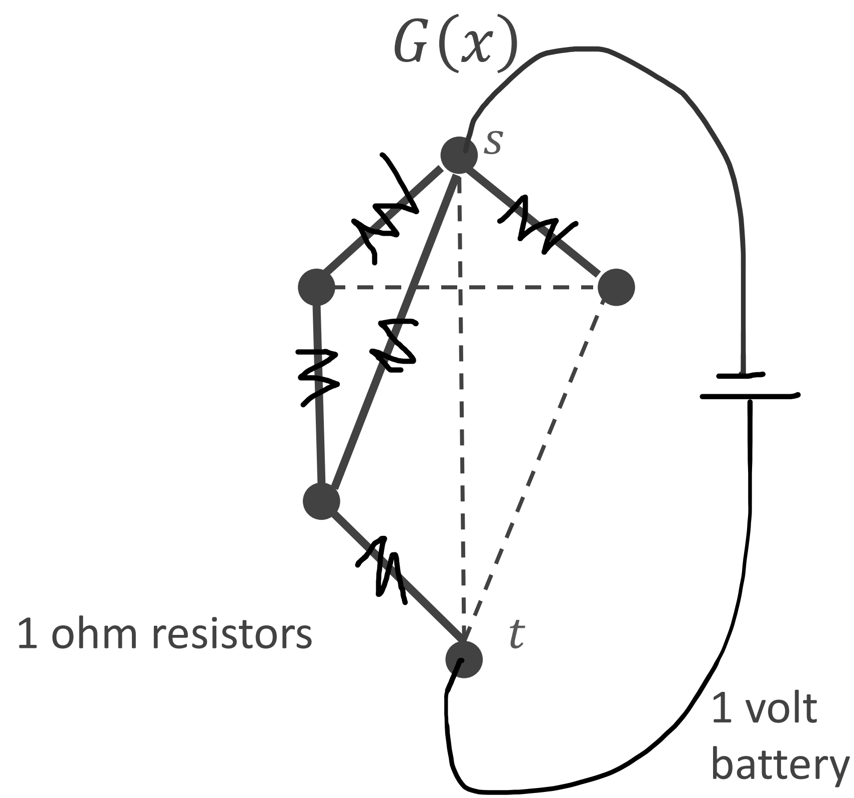
- effective resistance
- effective capacitance

Effective Resistance



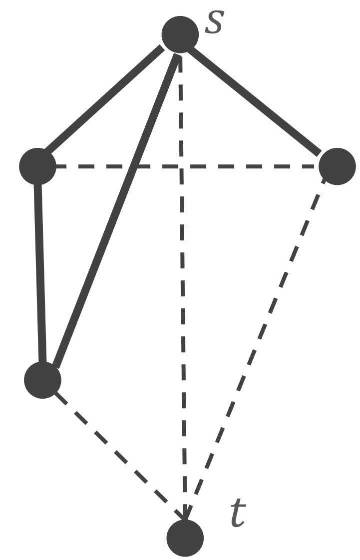
Effective Resistance

$R(G(x))$ is effective resistance of this circuit



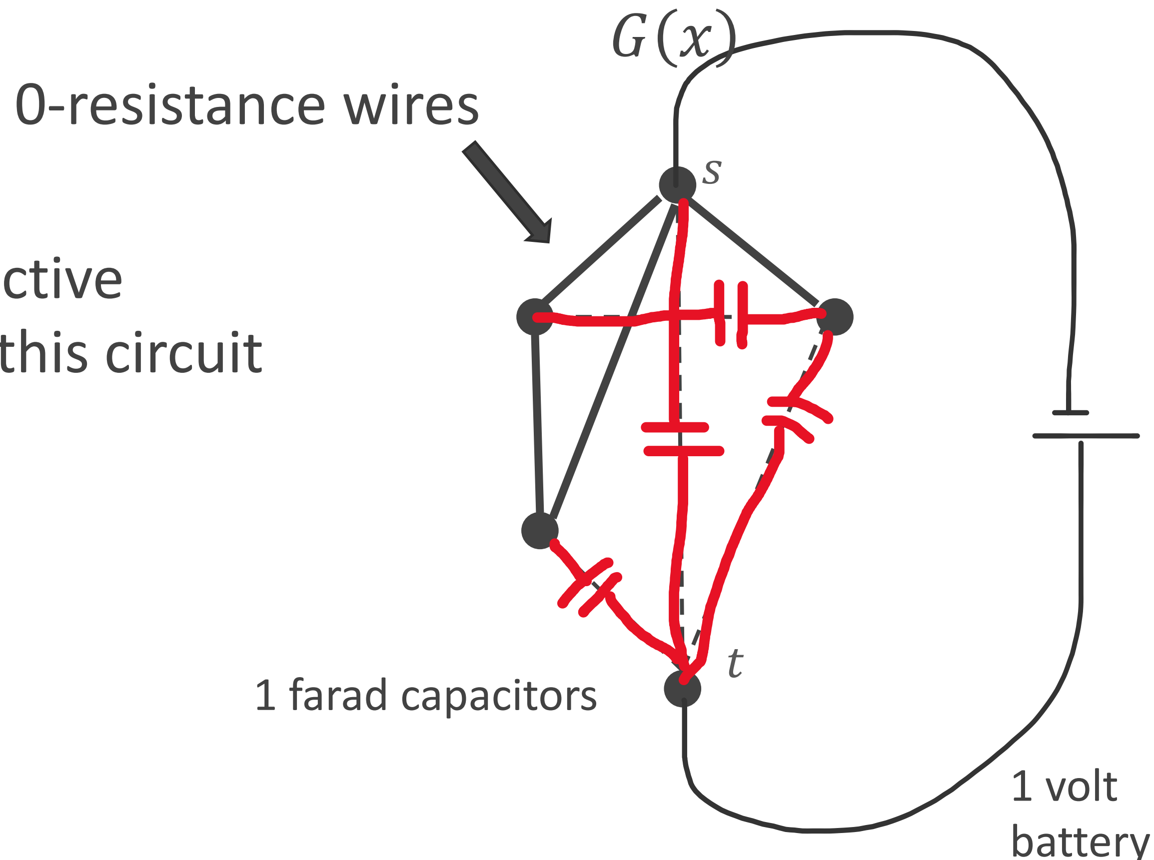
Effective Capacitance

$G(x)$



Effective Capacitance

$C(G(x))$ is effective capacitance of this circuit



Quantum Path Detection Algorithm Complexity

Quantum Query Complexity of Path Detection:

- If path in $G(x)$:

$$\tilde{O}\left(\sqrt{R\big(G(x)\big) \times \max_{\substack{y:G(y) \\ has\ no\ path}} C\big(G(y)\big)}\right)$$

[Belovs, Reichardt ’12; JJ**K**P, `18; AC**K**KY ’20]

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$$\tilde{O}\left(\sqrt{C\big(G(x)\big)\times \max_{\substack{y:G(y)\\ has\ a\ path}} R\big(G(y)\big)}\right)$$

[Belovs, Reichardt ’12; JJ**K**P, `18; AC**K**KY ’20]

Let's Design a Quantum Algorithm!

Search: Is there a 1 in $x = x_1 x_2 \ldots x_n$?

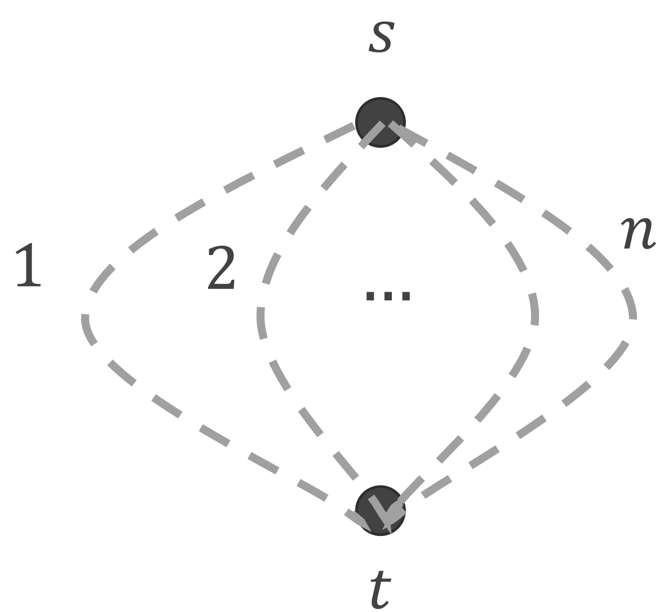
Let's Design a Quantum Algorithm!

Search: Is there a 1 in $x = x_1 x_2 \ldots x_n$?

Is there a 1 at bit 1?
OR
Is there a 1 at bit 2?
OR
Is there a 1 at bit 3?
OR
…

Let’s Design a Quantum Algorithm!

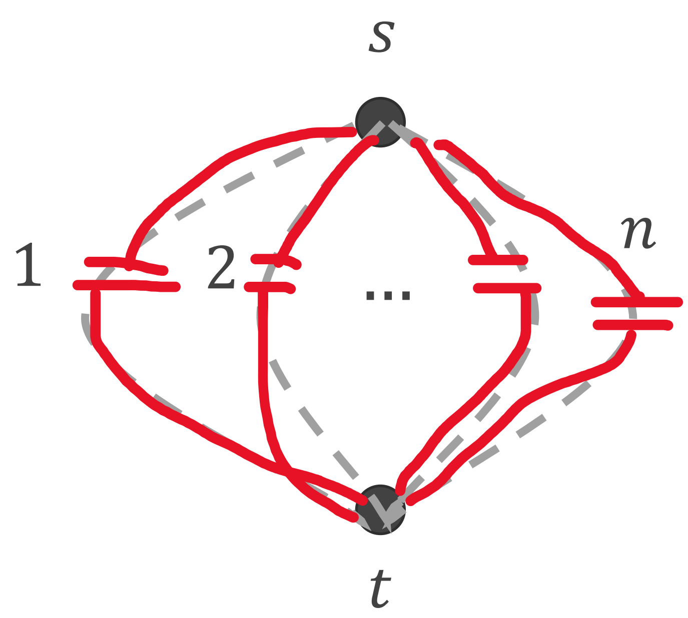
Search: Is there a 1 in $x = x_1 x_2 \ldots x_n$?



Let’s Design a Quantum Algorithm!

No path:

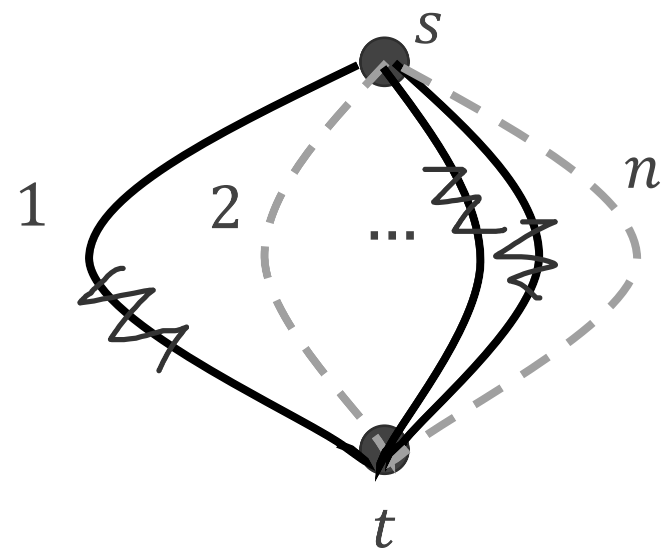
- $C(G(x)) = n$



Let's Design a Quantum Algorithm!

Path:

- $R(G(x)) = 1/k$ (k is number of bits with value 1)



Quantum Search Algorithm Complexity

Quantum Query Complexity of Path Detection:

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[Belovs, Reichardt ’12; JJ**K**P, `18; AC**K**KY ’20]

Quantum Search Algorithm Complexity

Quantum Query Complexity of Path Detection:

- If path in $G(x)$:

$$\tilde{O}\left(\sqrt{R(G(x)) \times \max_{\substack{y:G(y) \\ has\ no\ path}} C(G(y))}\right) = \tilde{O}\left(\sqrt{\frac{n}{k}}\right)$$

- If no path in $G(x)$:

$$\tilde{O}\left(\sqrt{C(G(x)) \times \max_{\substack{y:G(y) \\ has\ a\ path}} R(G(y))}\right) = \tilde{O}(\sqrt{n})$$

Not new result (variant of Grover's Algorithm), but you designed it! (Within log factors of optimal)

Performance

*There are alternative optimal algorithms for some problems

- Read-once Boolean formulas (query optimal) [Jeffery, **K,** ‘17]
- Total connectivity (query optimal) [Jarret, Jeffery, **K,** Piedrafita, ‘18]
- Cycle detection (query optimal) [Delorenzo, **K,** Witter, ‘19]
- Even length cycle detection [Delorenzo, **K,** Witter, ‘19]
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- st-connectivity (query optimal) (Reichardt, Belovs `12)
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Path Detection Algorithm:

$$U = R_1 R_2$$

Reflection that encodes quantum walk on the skeleton graph (Often $O(\log n)$ time, no queries)

Reflection that encodes where edges are in graph (Takes $O(1)$ time and $O(1)$ queries)

Path Detection Algorithm:

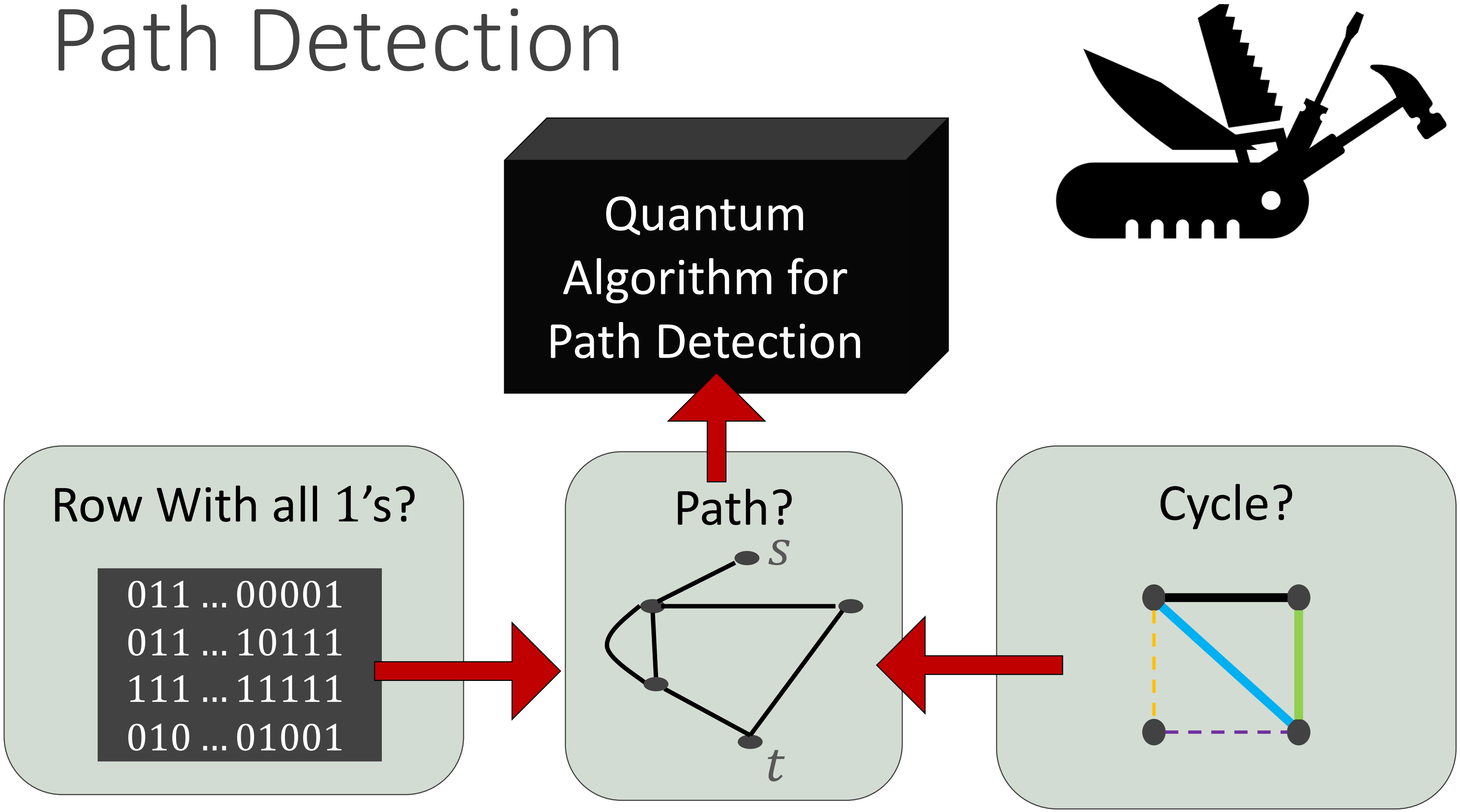
Basic Algorithm (no speed up for easy instances):

Do eigenvalue estimation of U for some initial state with high enough precision

- If initial state has eigenvalue 1 -> decide path
- Otherwise -> decide no path

[Belovs, Reichardt ’12]

Path Detection



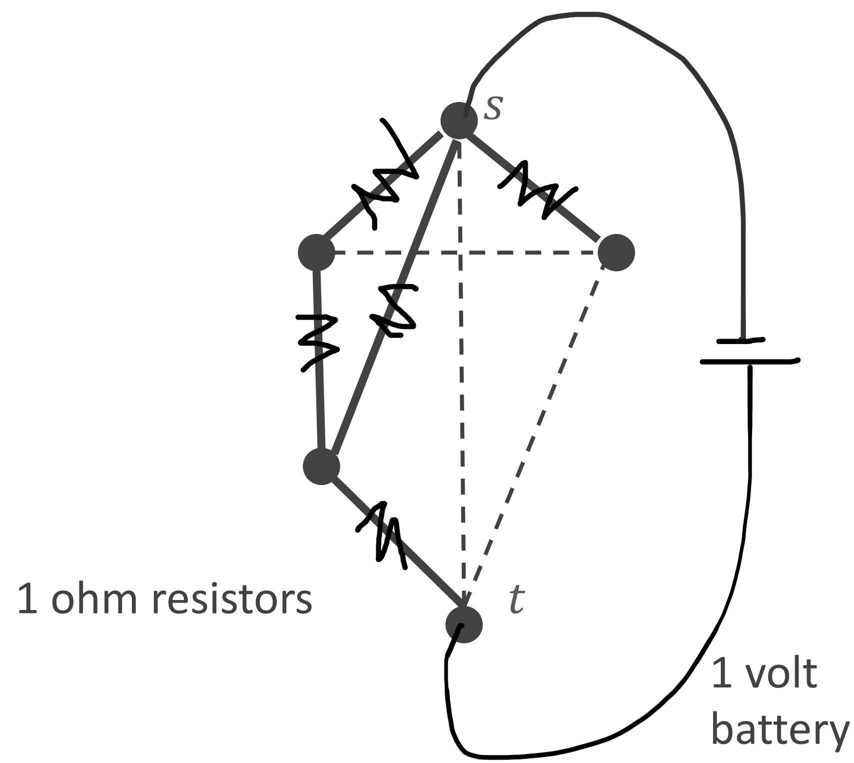
Reduce our problem to path detection

Future/Current Work

- Recent work (with Stacey Jeffery and Alvaro Piedrafita): algorithm to sample an edge in the flow

Effective Resistance

$R(G(x))$ is effective resistance of this circuit



Future/Current Work

- Recent work (with Stacey Jeffery and Alvaro Piedrafita): algorithm to sample an edge in the flow
  - Can find a path faster than existing algs for graphs with only short paths
  - Can figuring out how to sabotage connections in graphs with high throughput bridges
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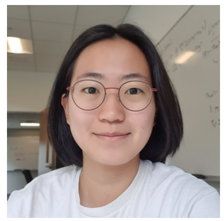
  - Other applications?
- When is path detection reduction approach optimal?
- Way to find optimal weights?

Thank you!

Funding:

ARO

Collaborators:



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