Path Detection:
A Quantum Computing Primitive

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Based on work with
Stacey Jeffery: arXiv: 1704.00765 (Quantum vol 1 p 26)
Michael Jarret, Stacey Jeffery, Alvaro Piedrafita, *in progress*

Middlebury

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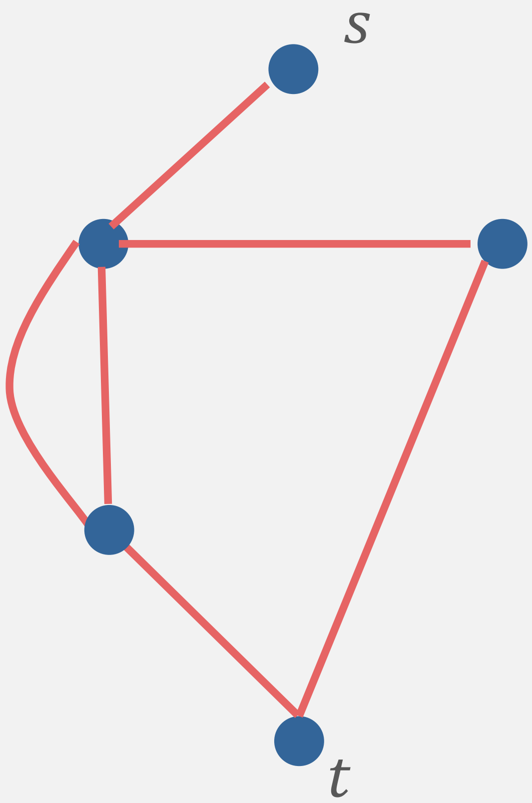
- New primitive: ***st*-connectivity**

Outline:

A. Introduction to st-connectivity
B. st-connectivity makes a good algorithmic primitive
   1. Applies to a wide range of problems
   2. Easy to understand (without knowing quantum mechanics)
C. Applications and performance of algorithm

st-connectivity

$\boldsymbol{st-connectivity}$:
is there a path from s to t?



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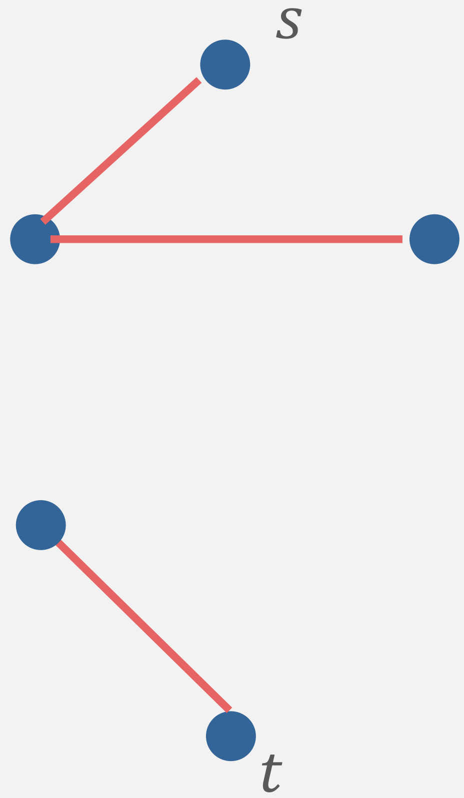
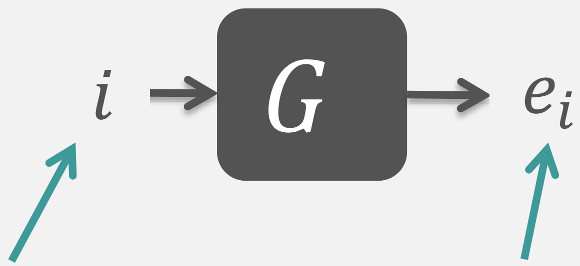
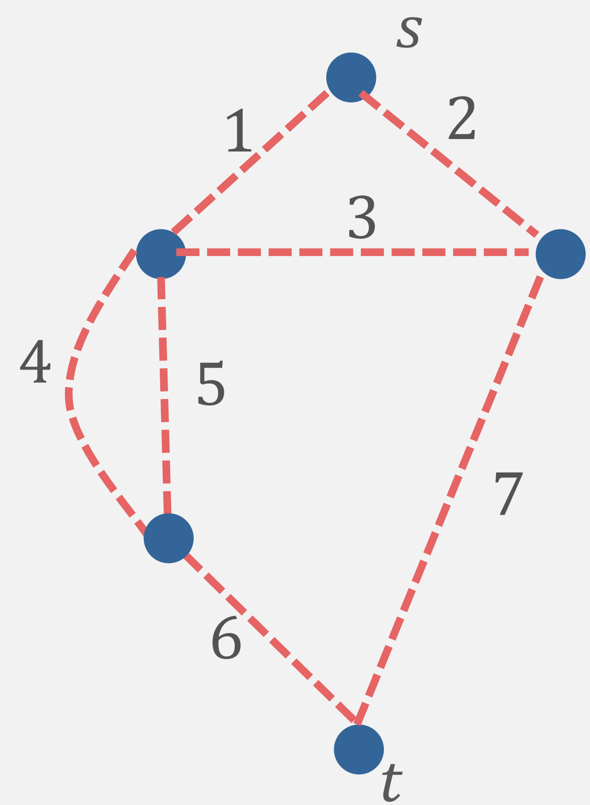


Black Box Model



Edge label

- $e_i = 1$ if i^{th} edge is there
- $e_i = 0$ if edge is not there



Let $\mathcal{H}$ be the set of graphs G that the black box might contain.

Figure of Merit

- Query Complexity
  - Number of uses (queries) of the black box
  - All other operations are free

- Under mild assumption, for our algorithm,

  quantum query complexity ≅ quantum time complexity

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A. Introduction to st-connectivity
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        - Evaluating Boolean formulas reduces to st-connectivity
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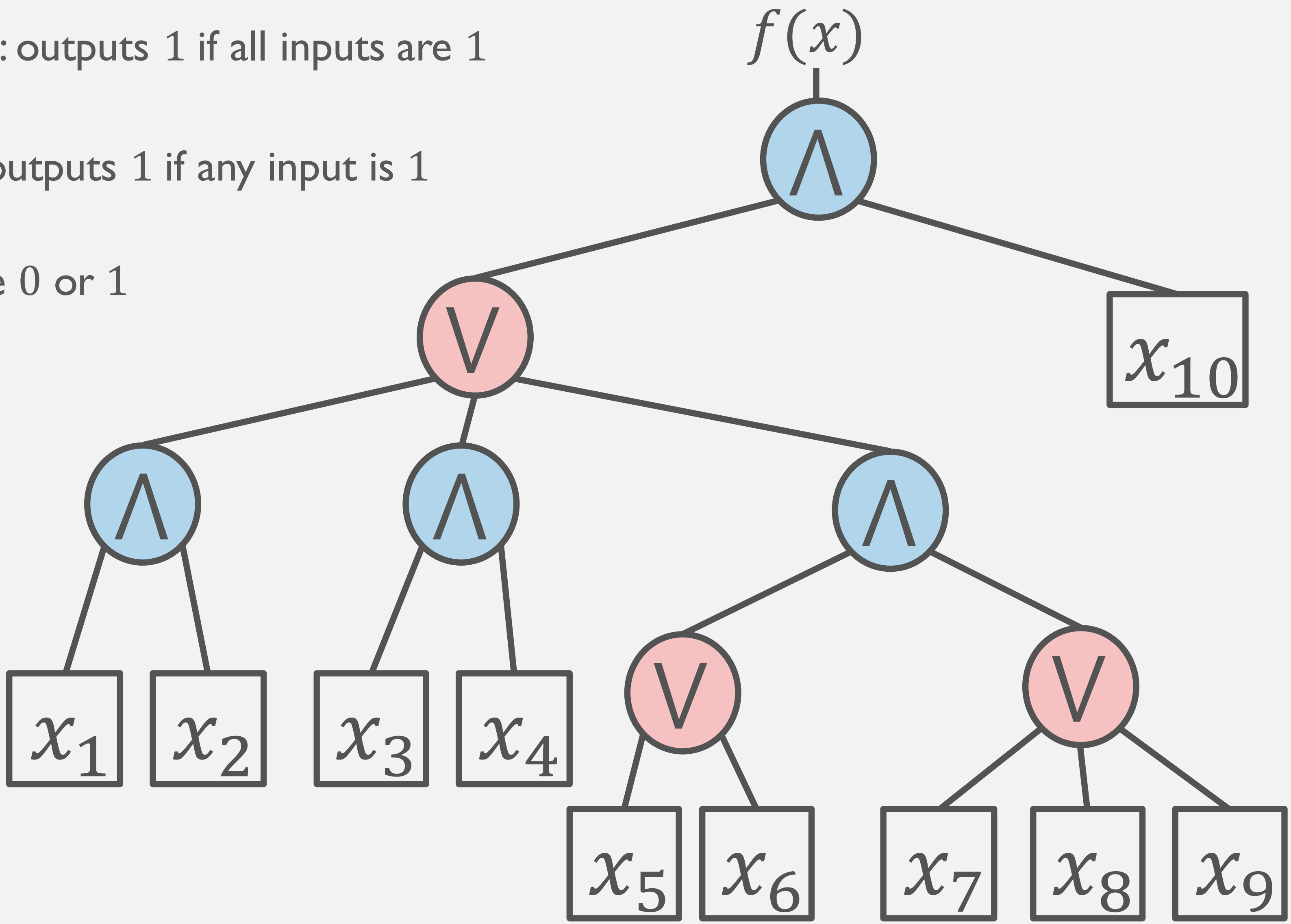
Boolean Formulas

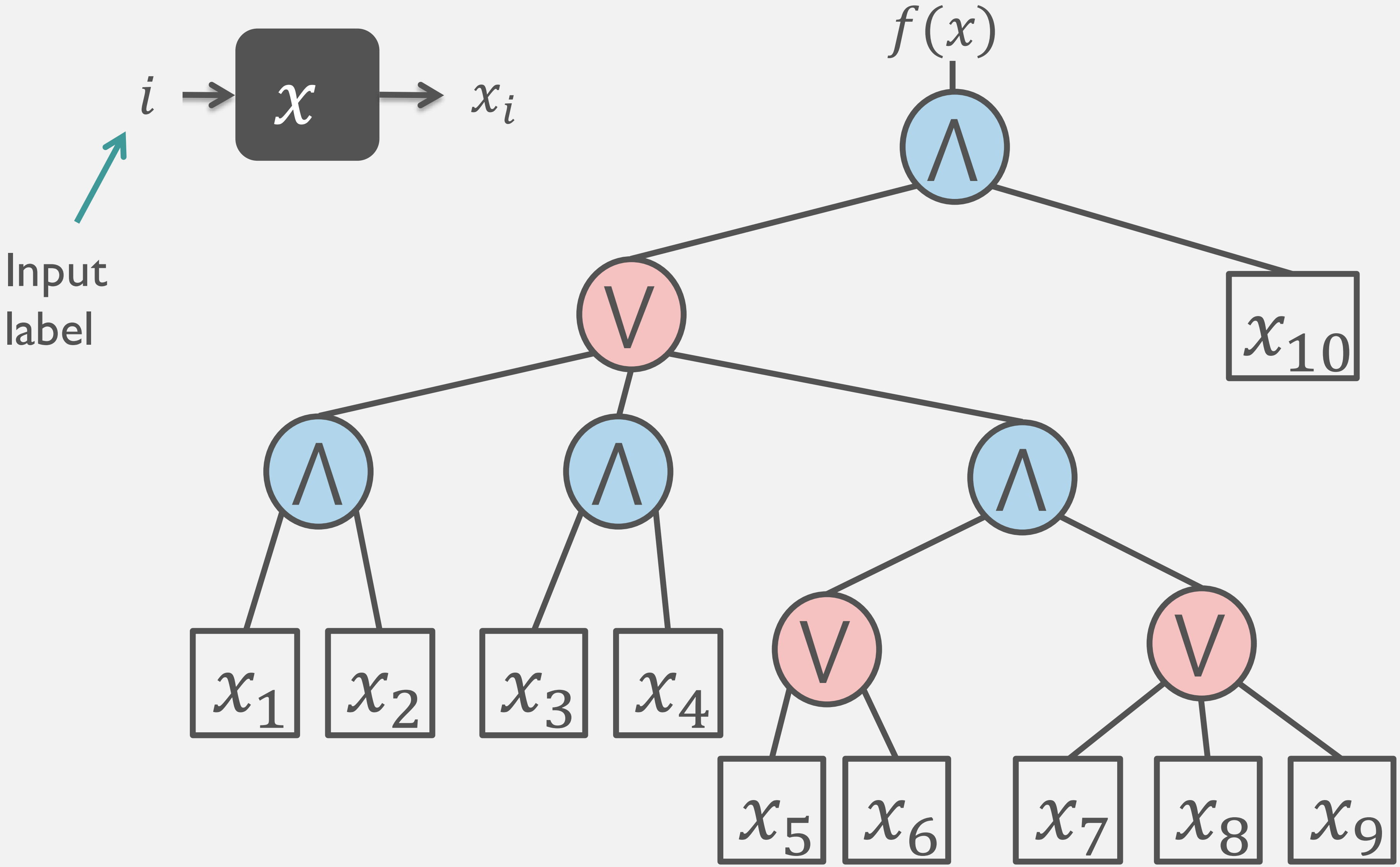
$\wedge$ AND: outputs 1 if all inputs are 1

$\vee$ OR: outputs 1 if any input is 1

x_i Value 0 or 1

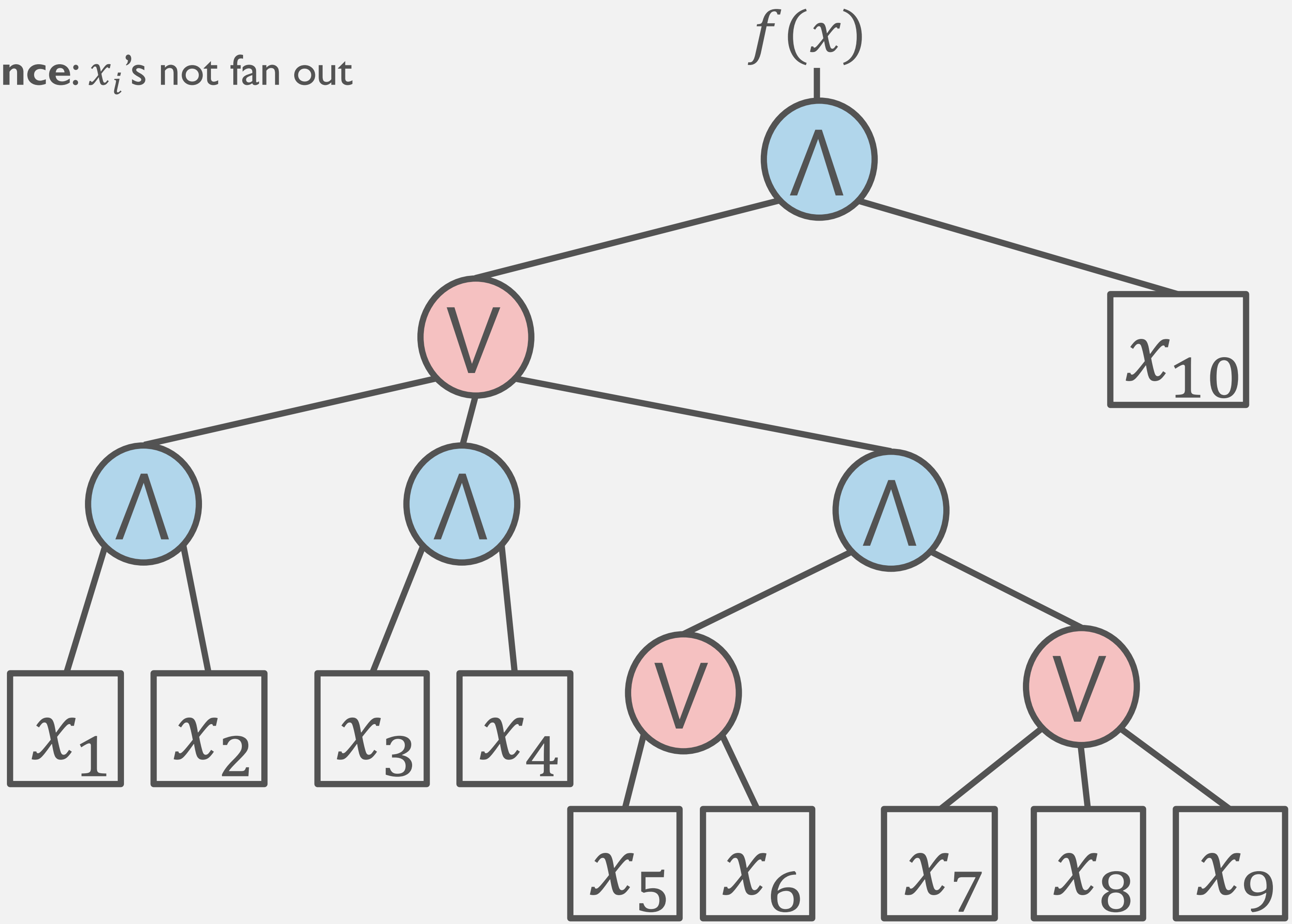


Boolean Formulas



Boolean Formulas

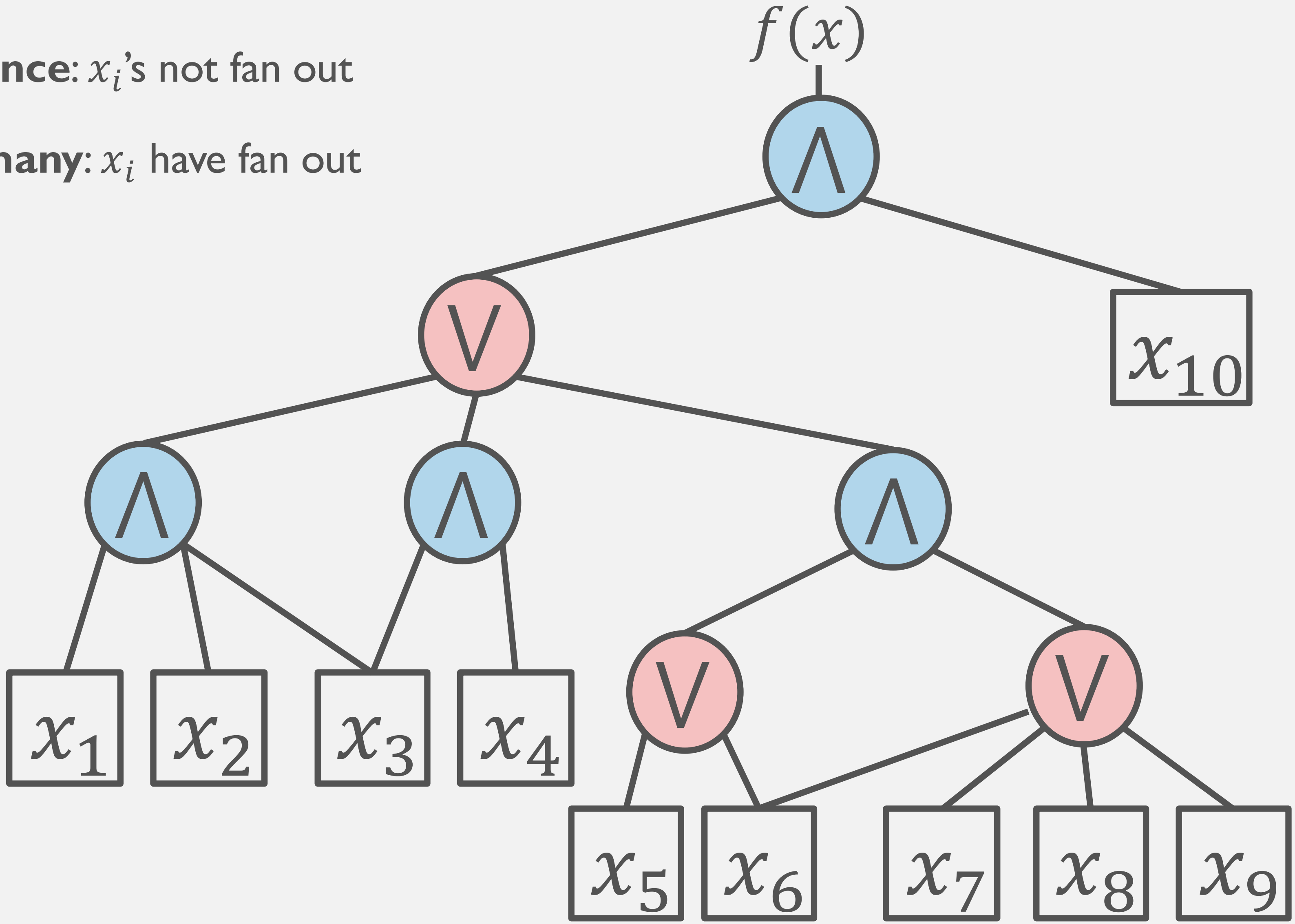
Read-once: x_i's not fan out



Boolean Formulas

Read-once: x_i's not fan out

Read-many: x_i have fan out

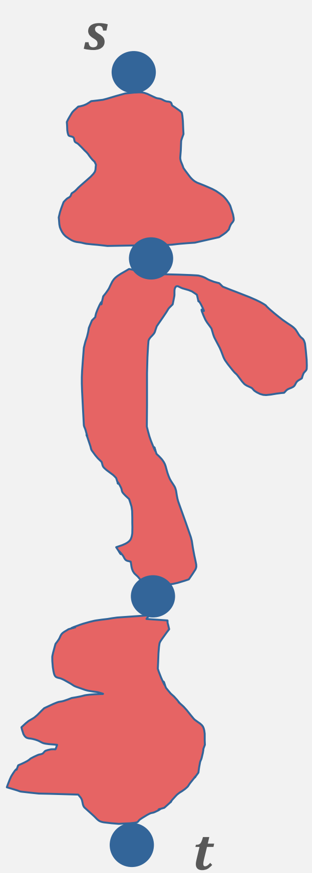


Boolean Formula Applications

- Logic
- Designing electrical circuits
- Game theory (deciding who will win a game)
- Combinatorics and graph problems
- Linear programming
- Testing potential solution to an NP-complete problem

Application to Boolean Formulas

$\wedge$ AND: outputs 1 if all input subformulas have value 1



s and t are connected if all subgraphs are connected

Application to Boolean Formulas

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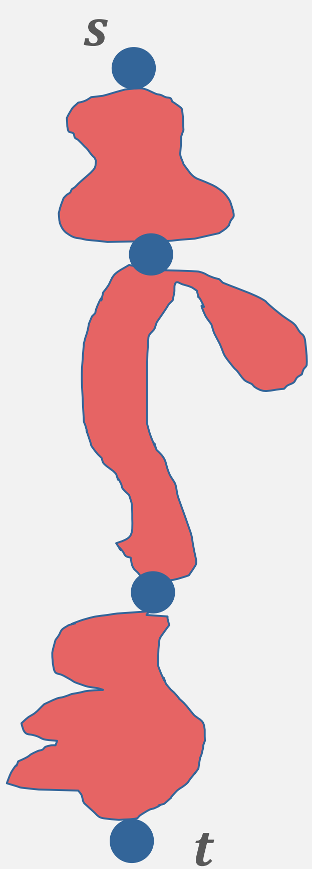

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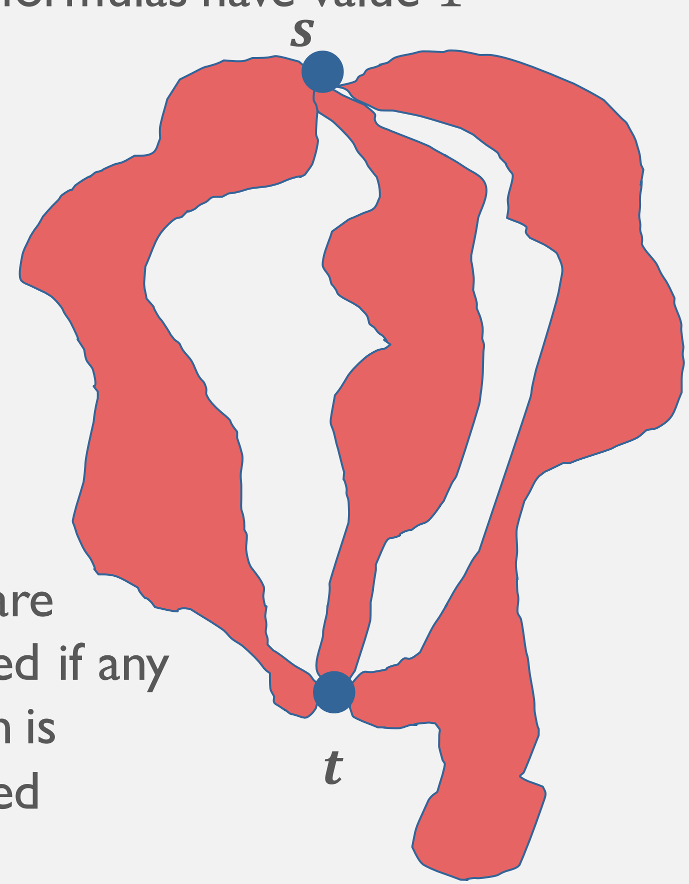

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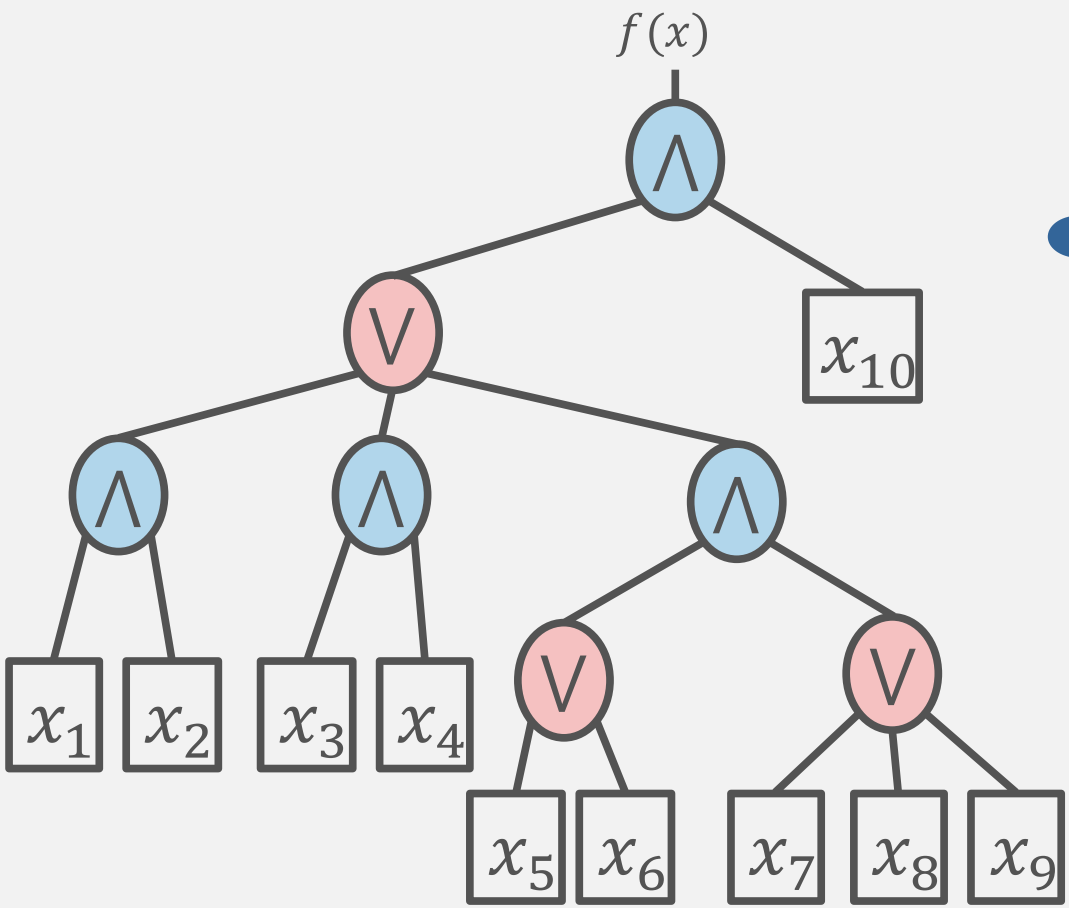
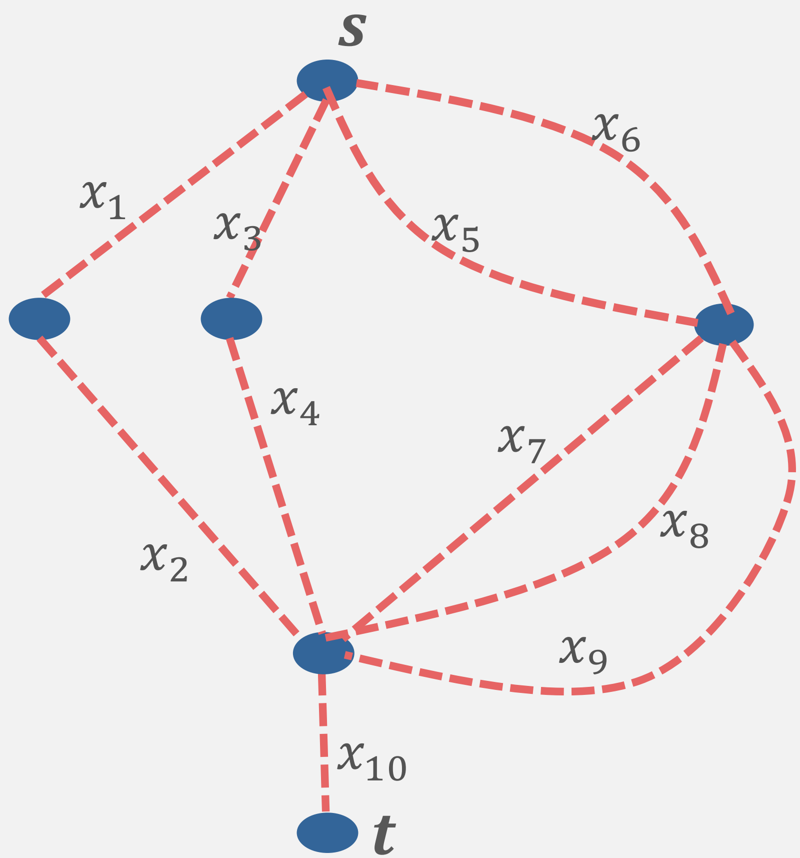
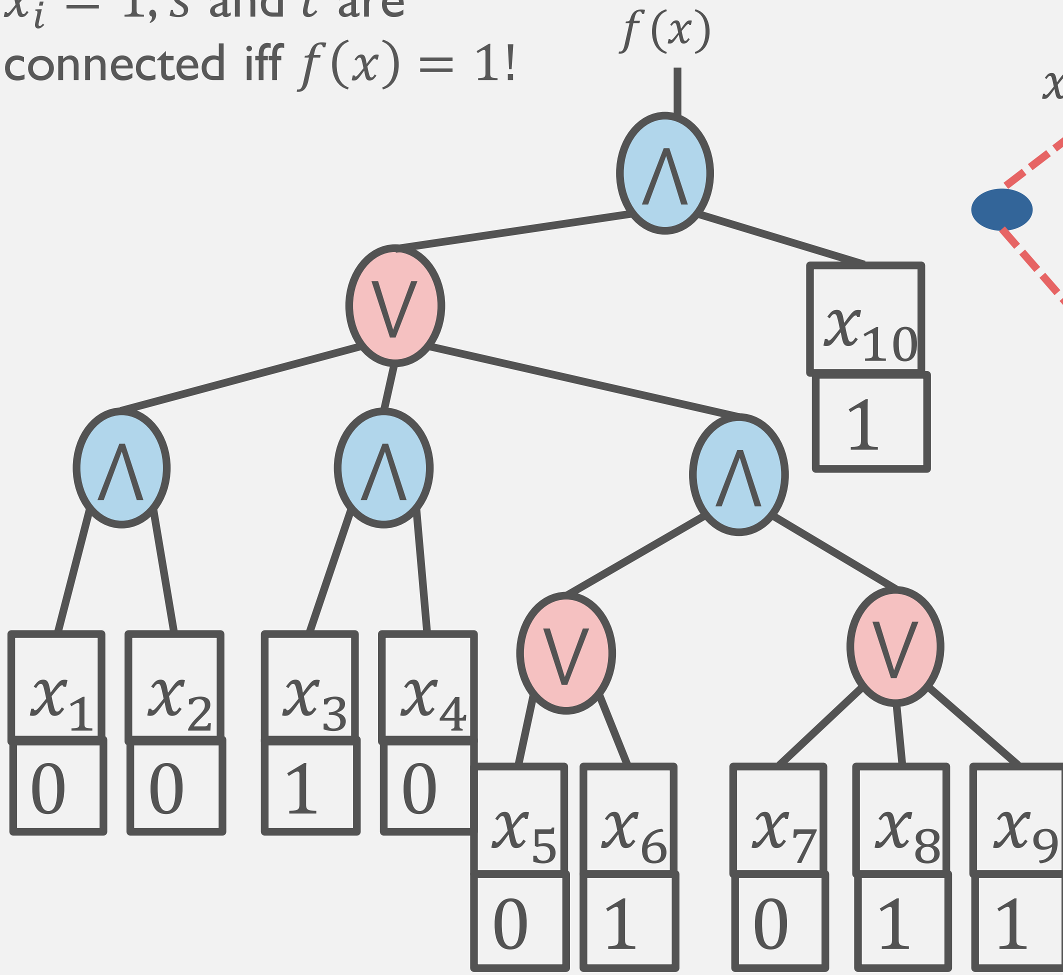
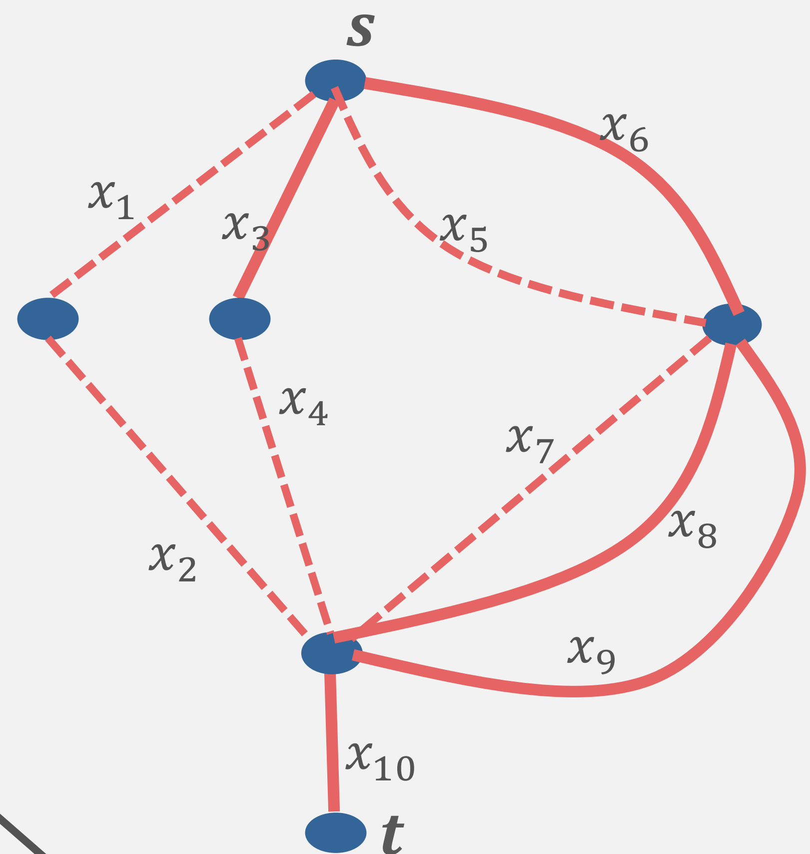




Application to Boolean Formulas

- If we put edges where $x_i = 1$, s and t are connected iff $f(x) = 1$!





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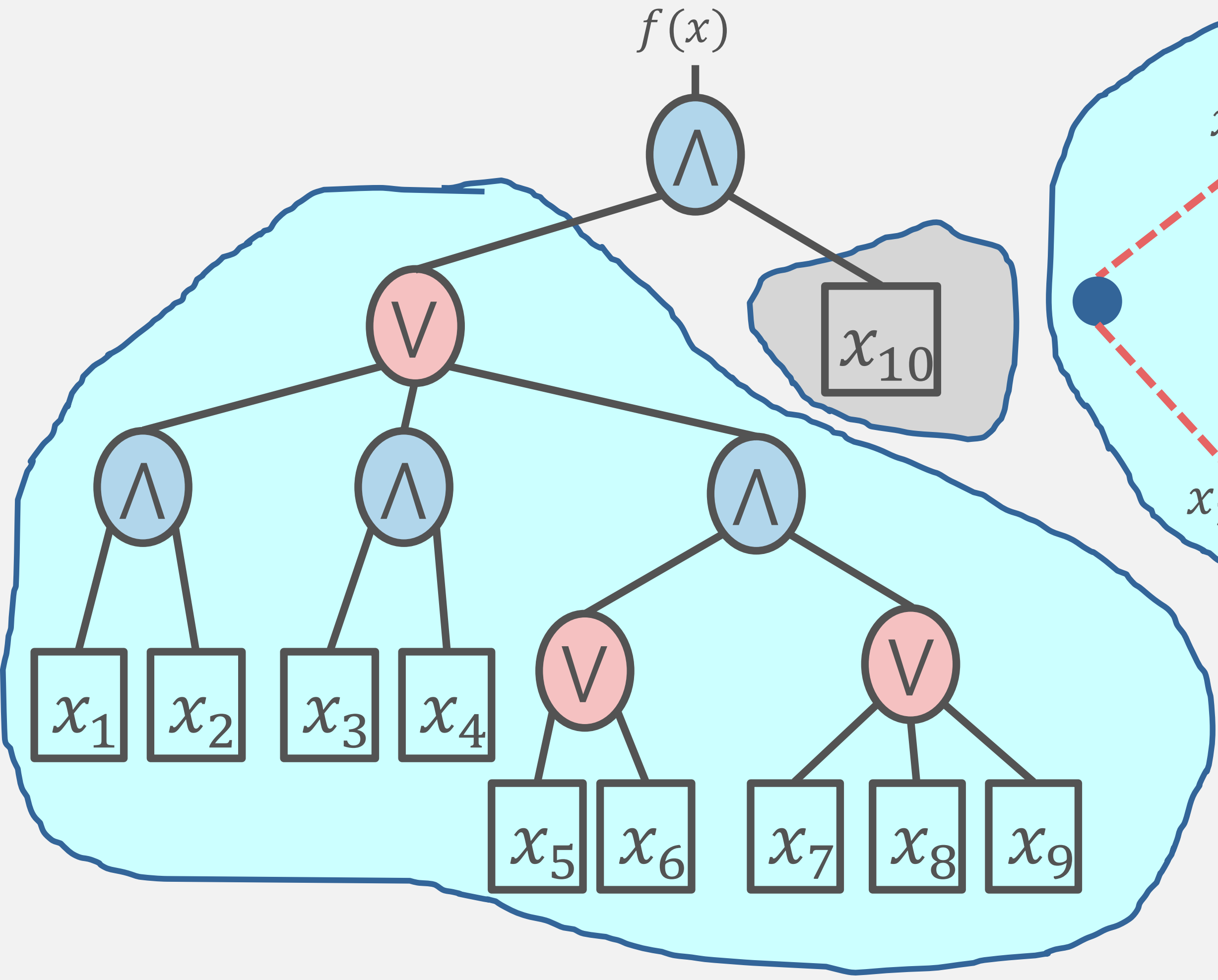

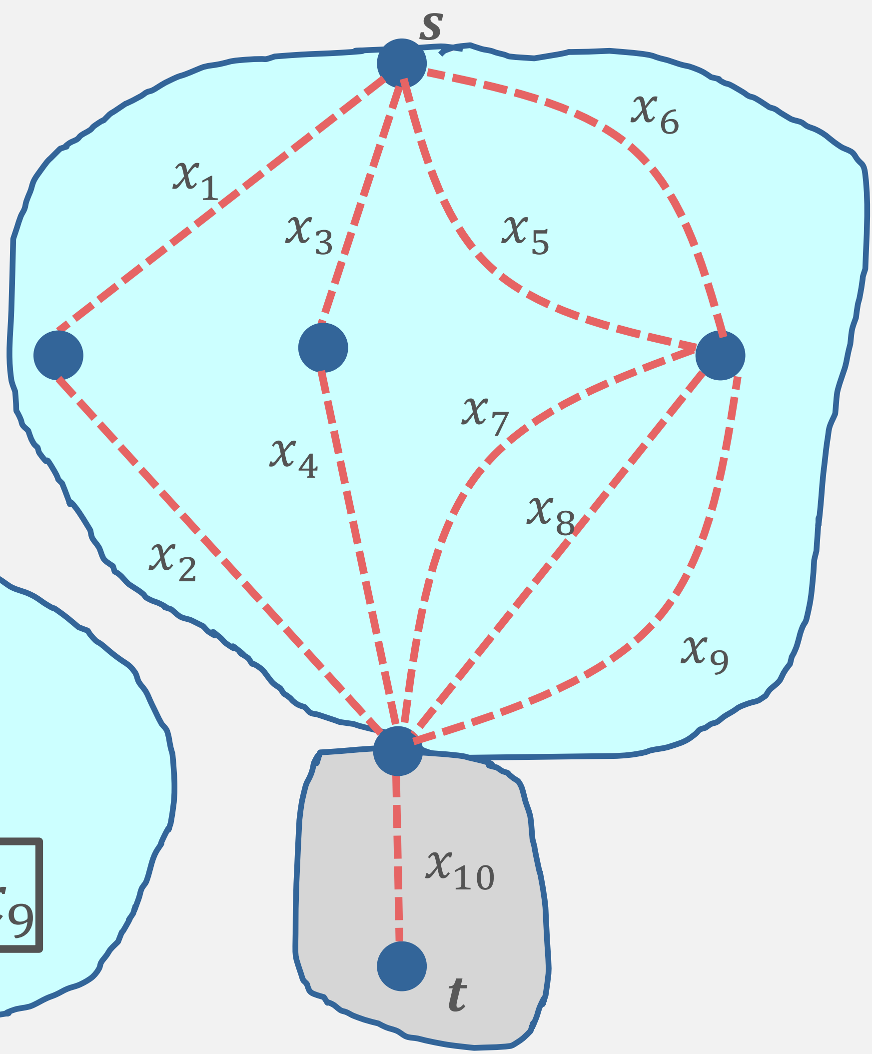

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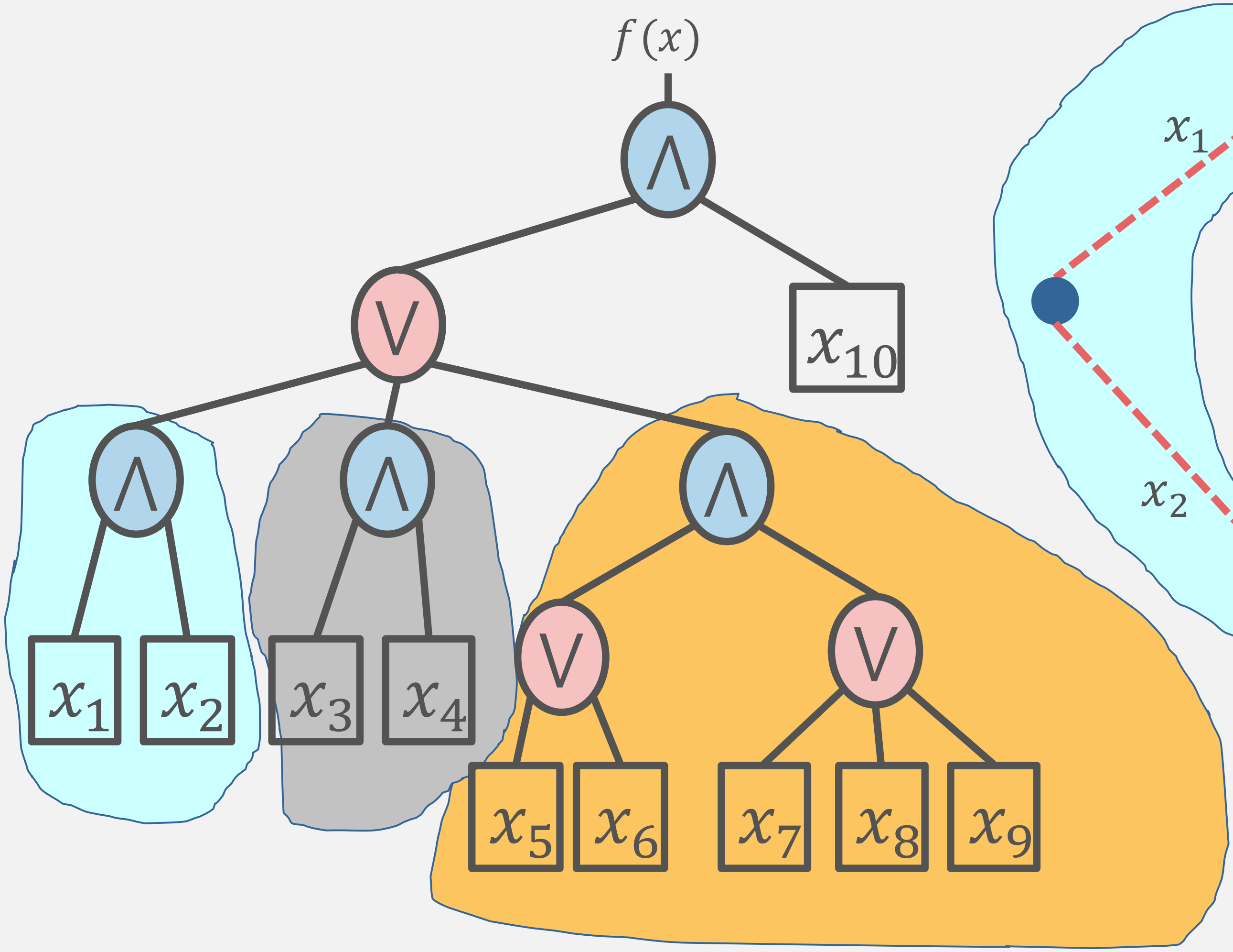

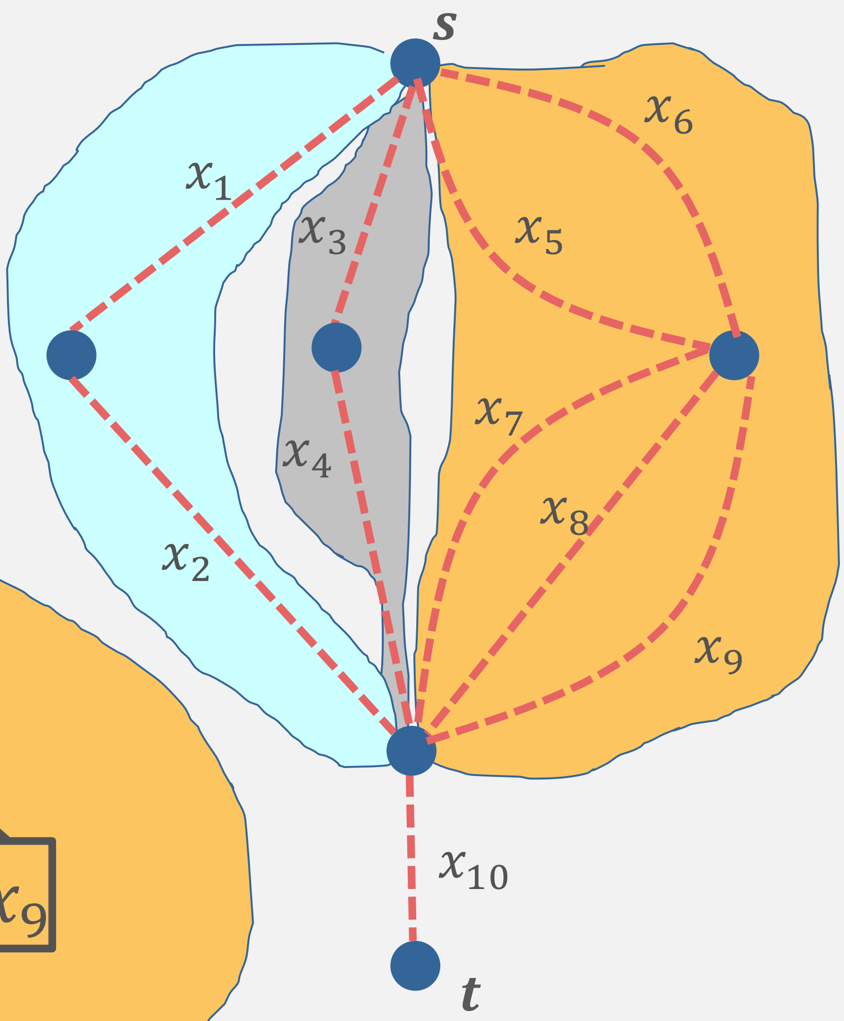

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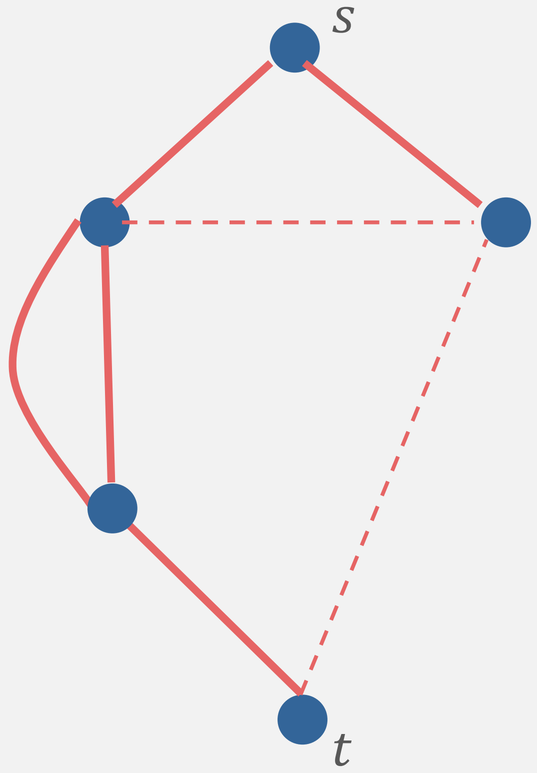
Effective Resistance

Graph G:



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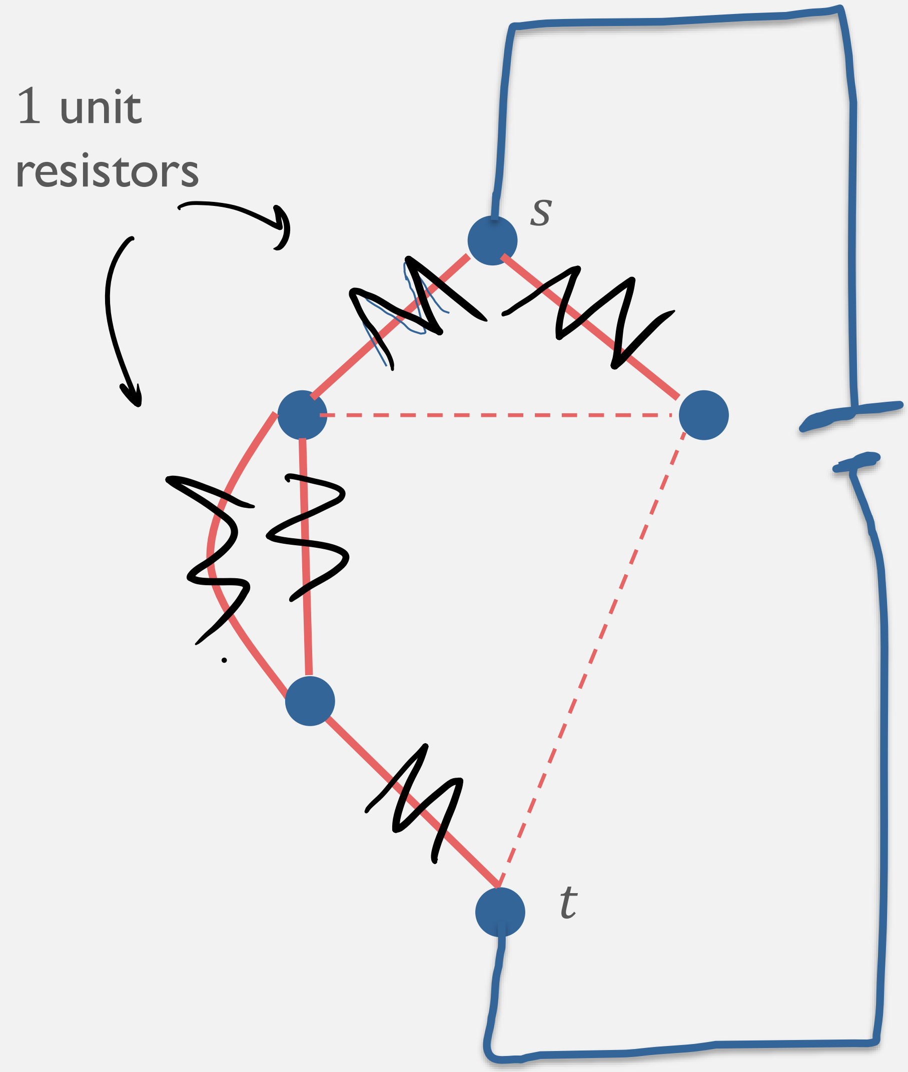

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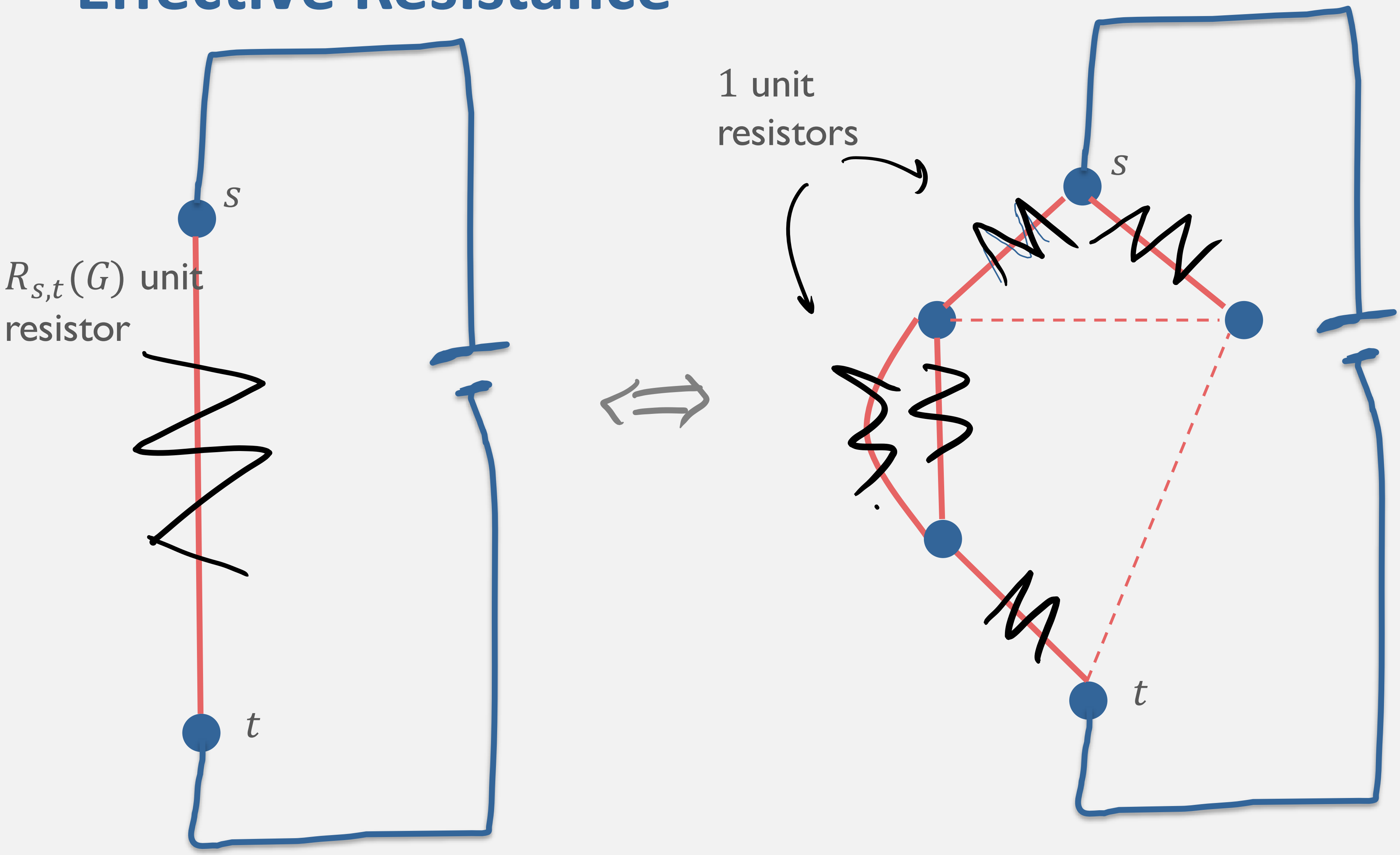

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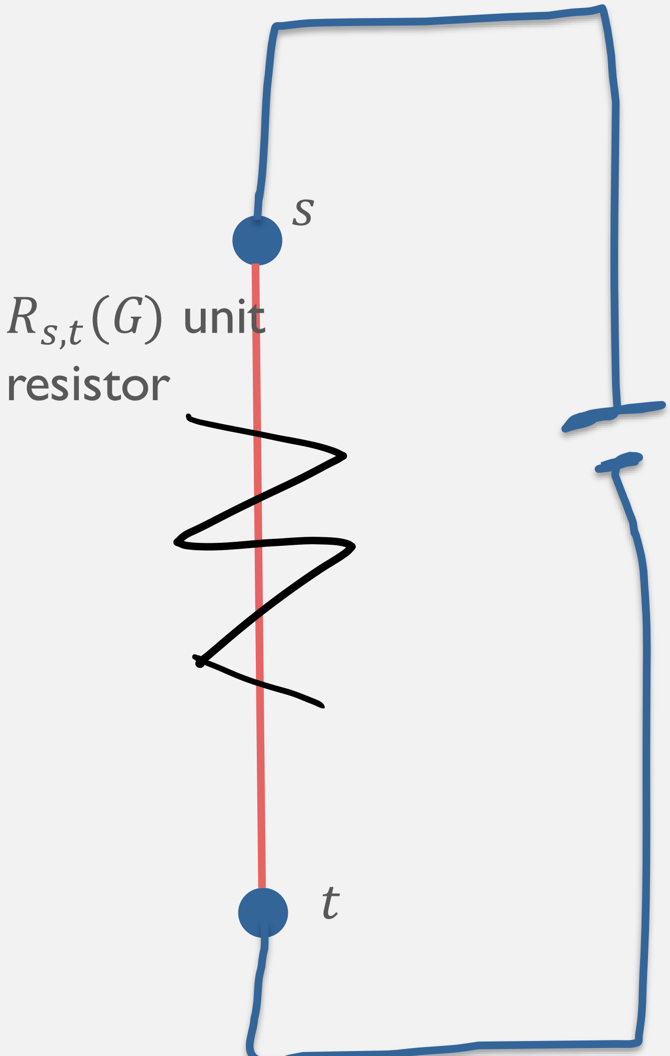
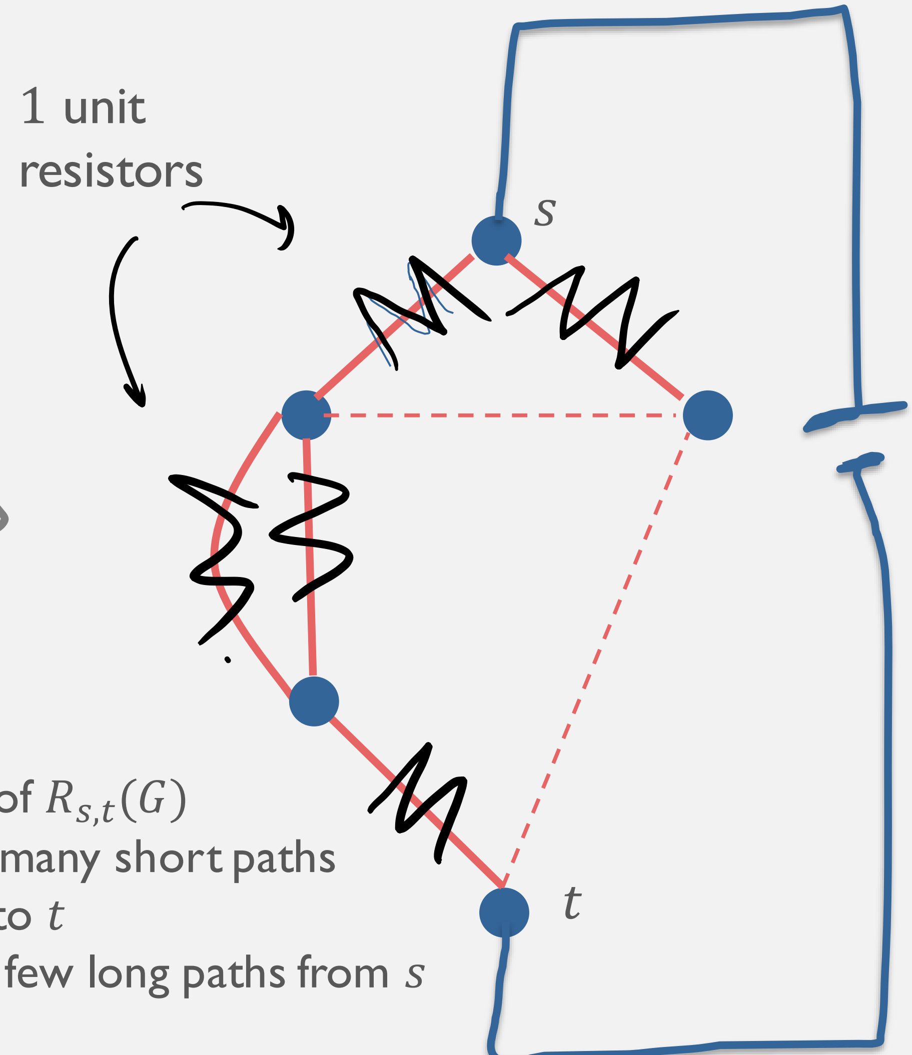


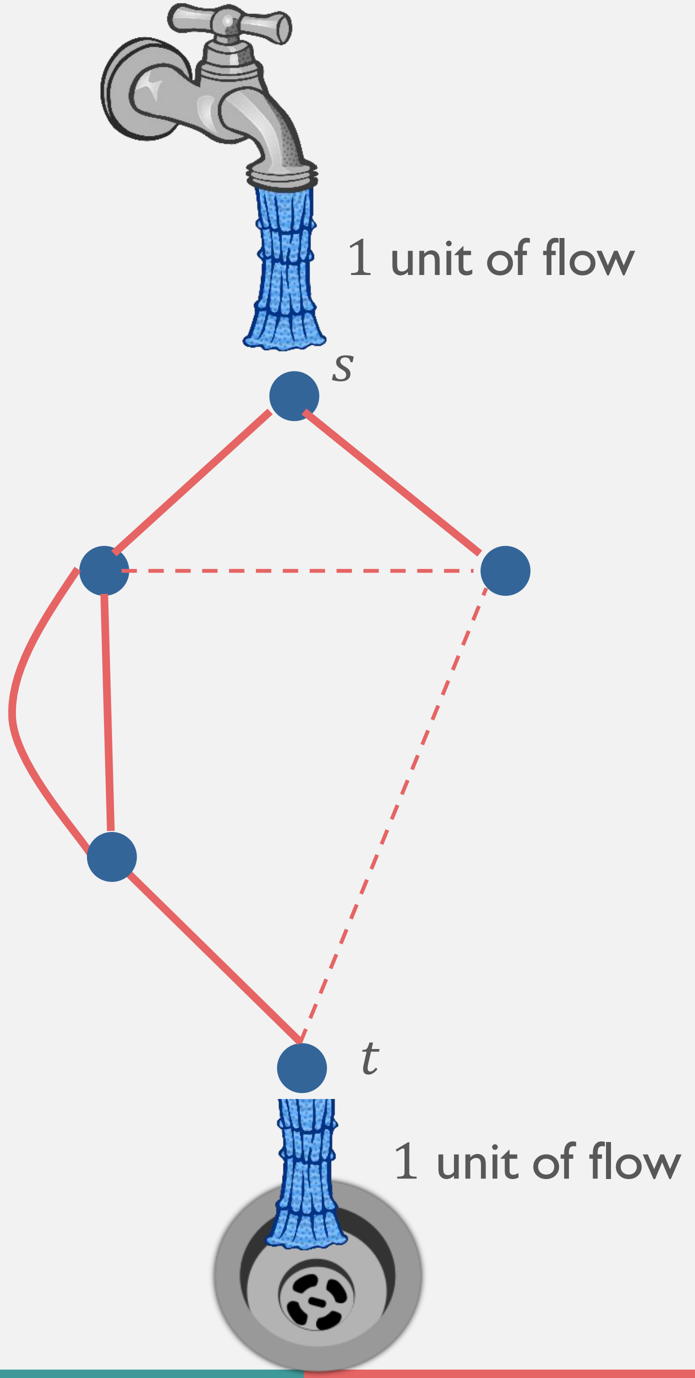
$\Longleftrightarrow$



Properties of $R_{s,t}(G)$

- Small if many short paths from s to t
- Large if few long paths from s to t
- Infinite if s and t not connected

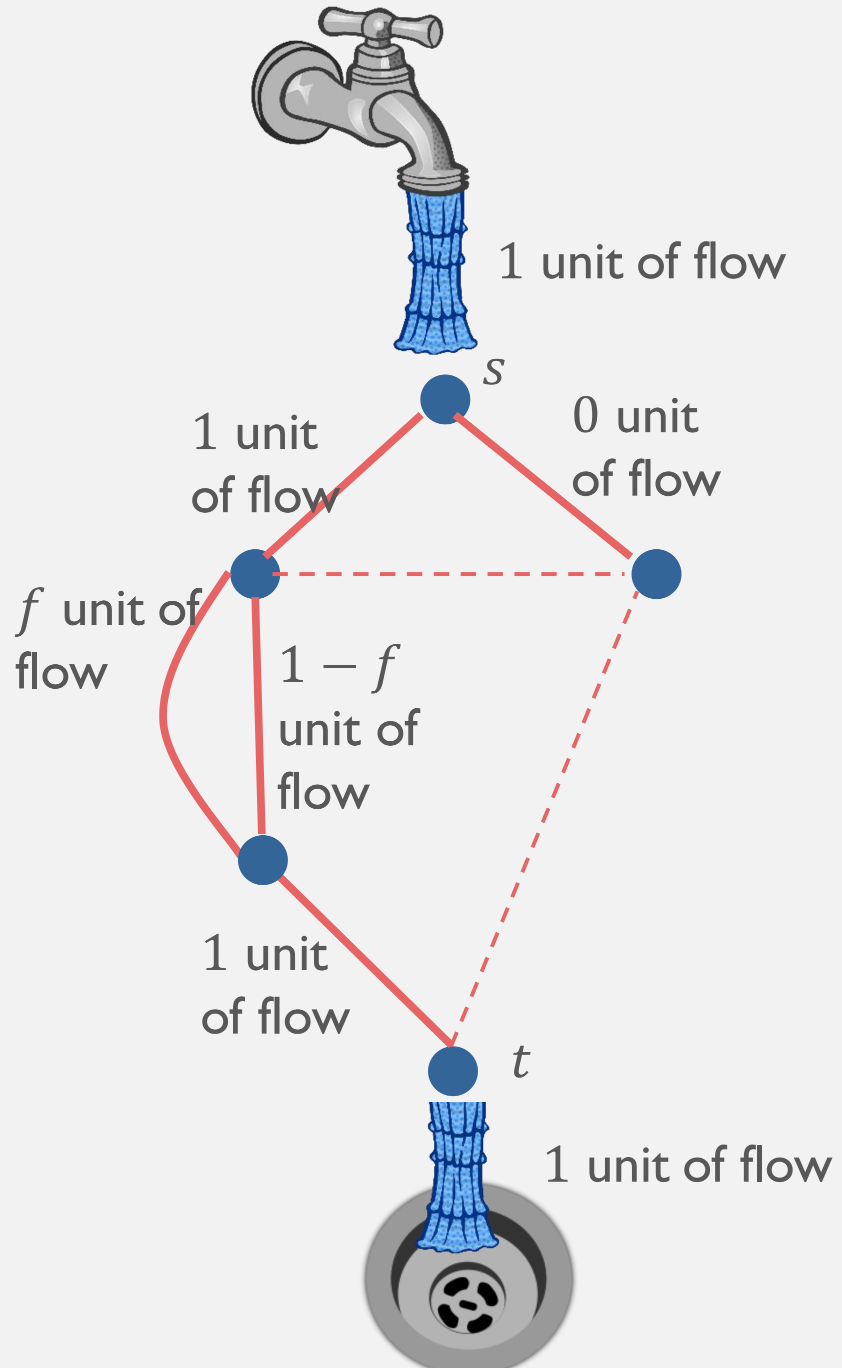
Effective Resistance



Effective Resistance

Valid flow:

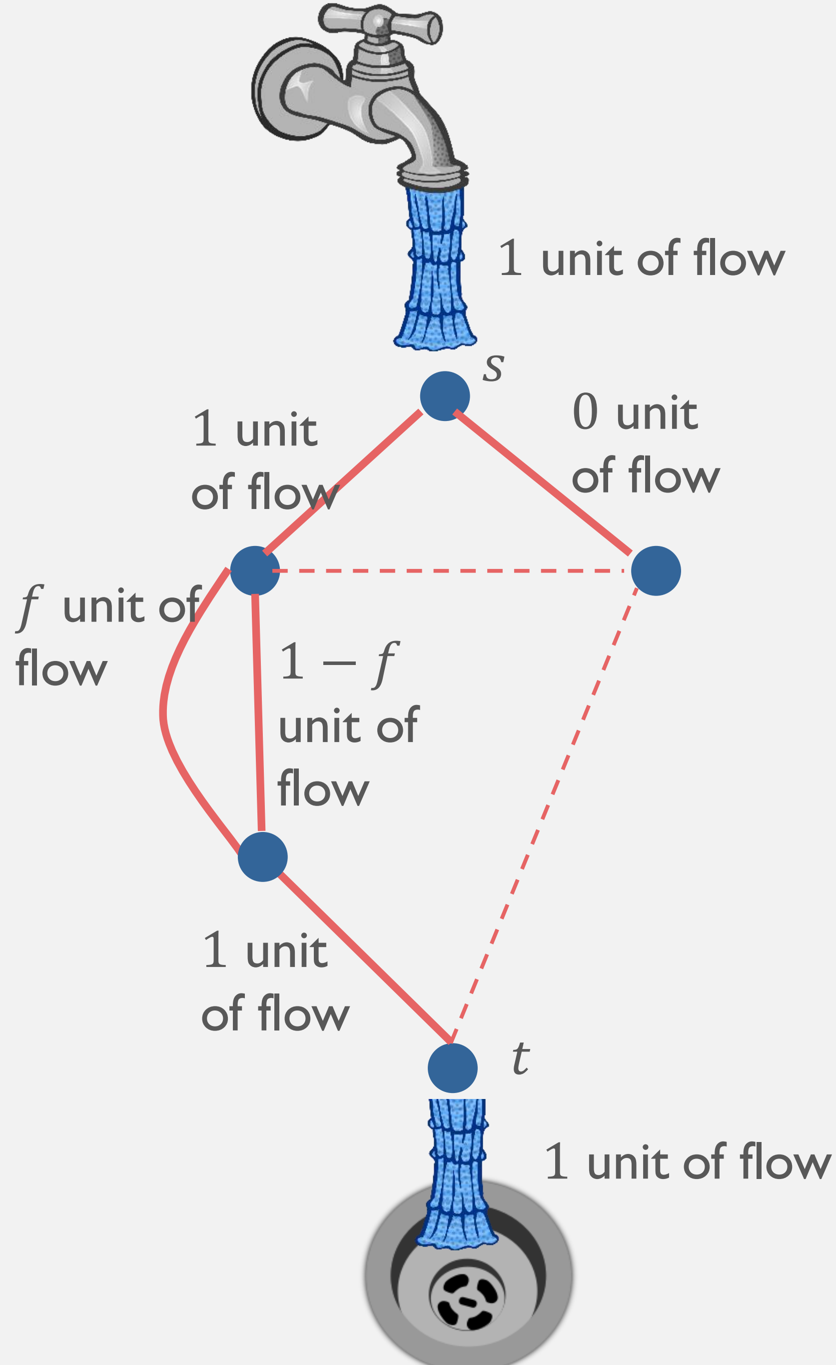
- 1 unit in at s
- 1 unit out at t
- At all other nodes, zero net flow



Effective Resistance

Flow energy:

$$\sum_{edges} (flow\ on\ edge)^2$$



Effective Resistance

Flow energy:

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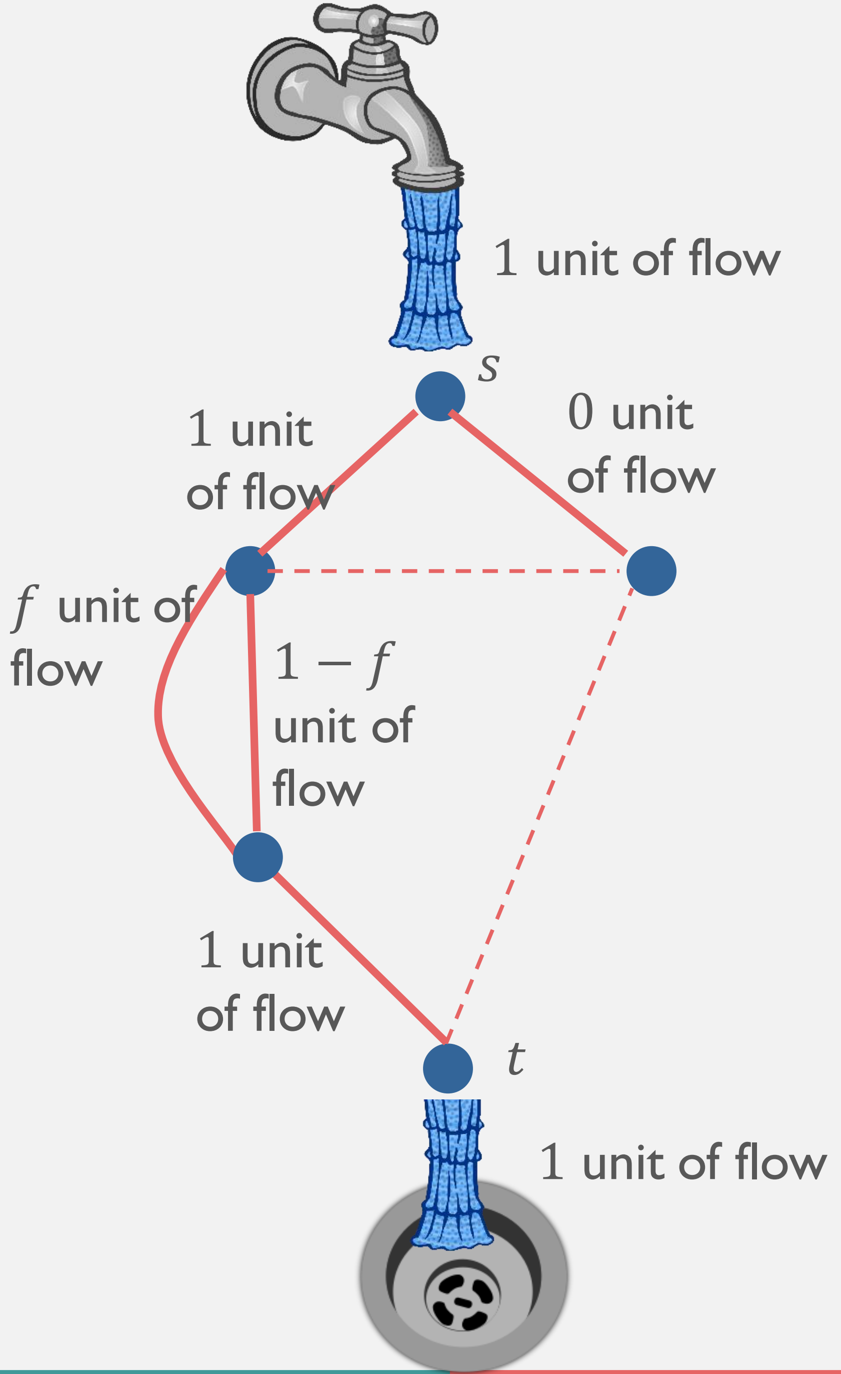
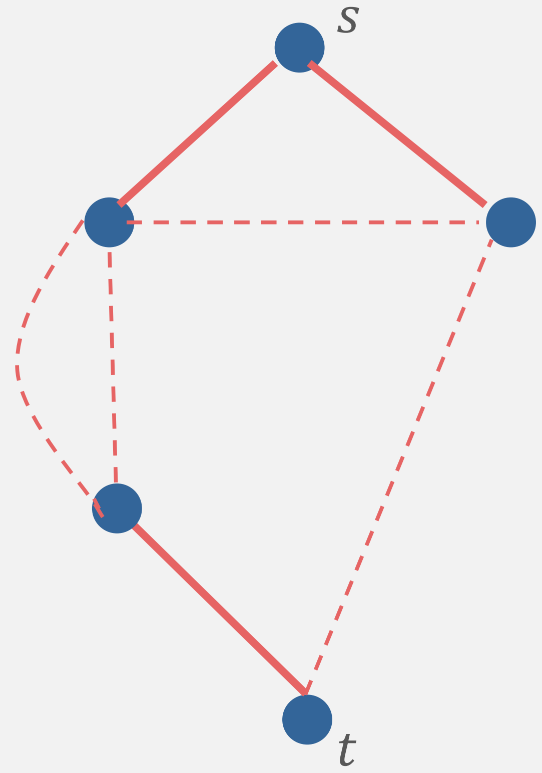
Effective Resistance: $R_{s,t}(G)$
Smallest energy of any valid flow from s to t on G.



Effective Capacitance

Graph G':



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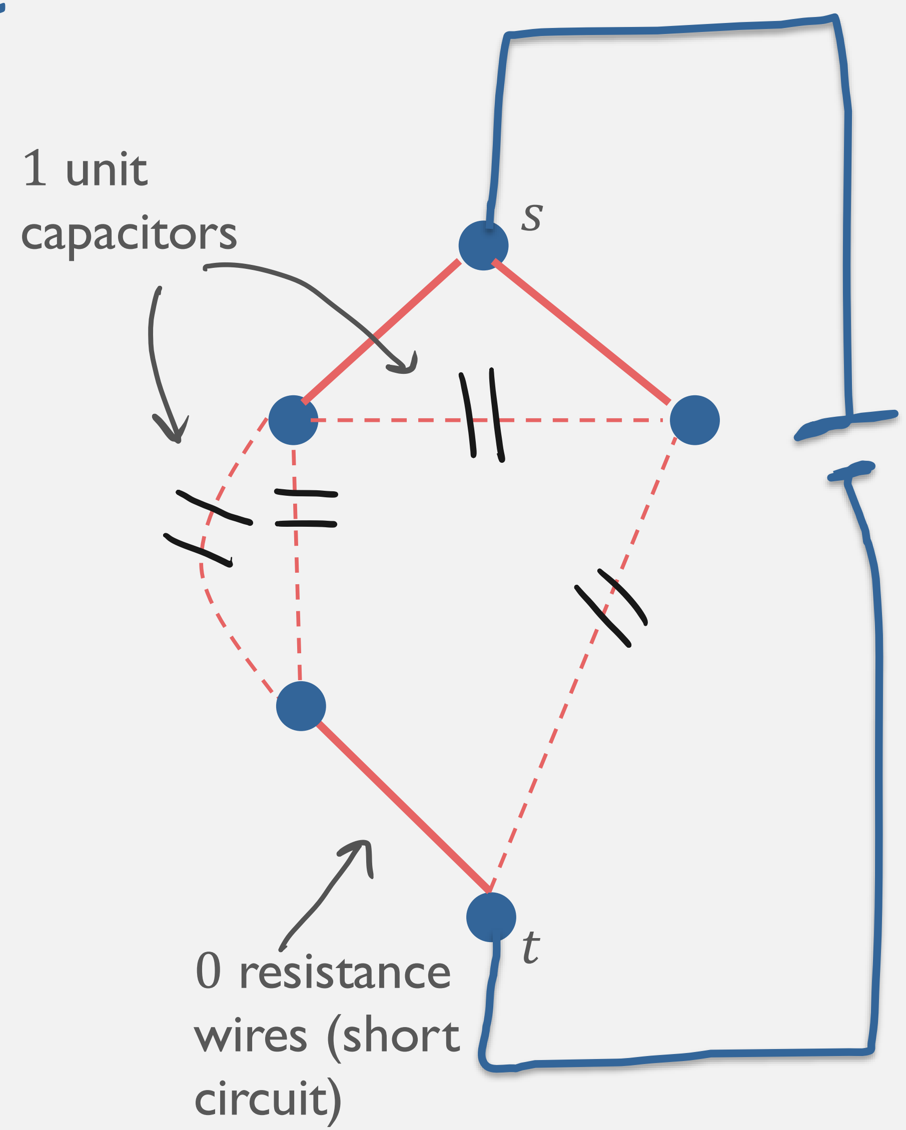

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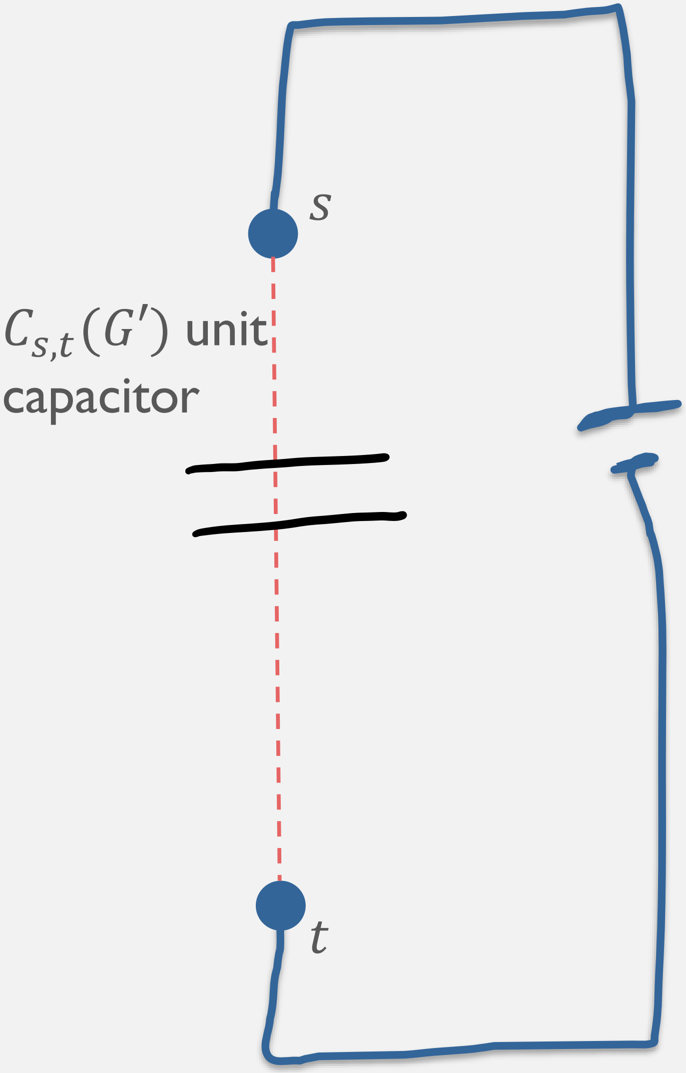

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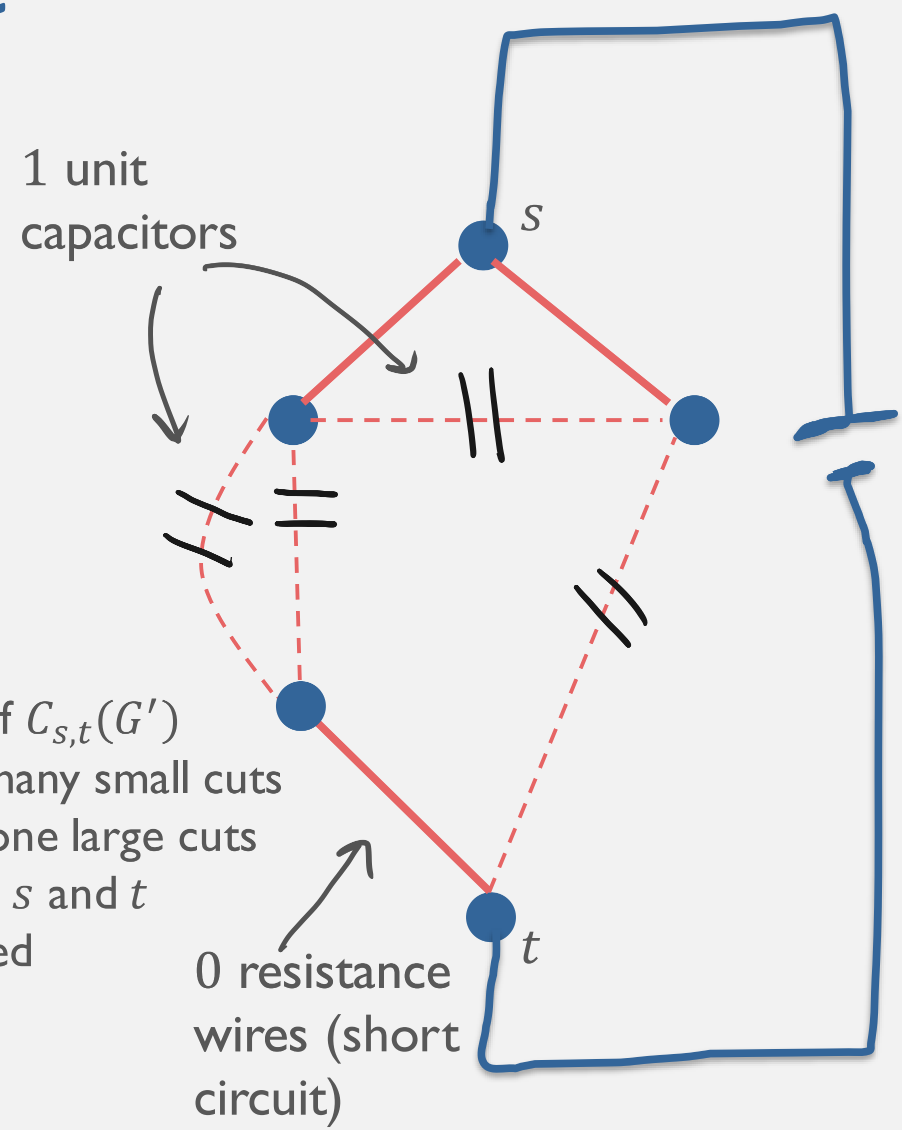


Properties of $C_{s,t}(G')$

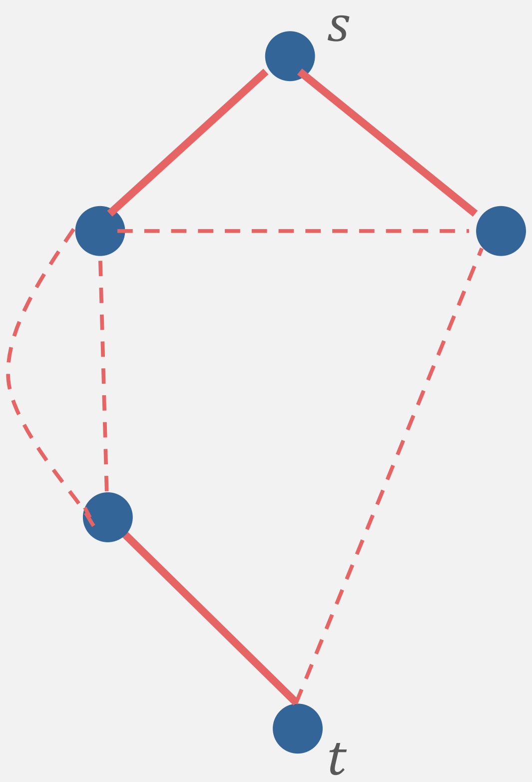
- Small if many small cuts
- Large if one large cuts
- Infinite if s and t connected

Effective Capacitance

Valid potential energy:

- 1 at s
- 0 at t
- Potential energy difference is 0 across edge

Graph G':



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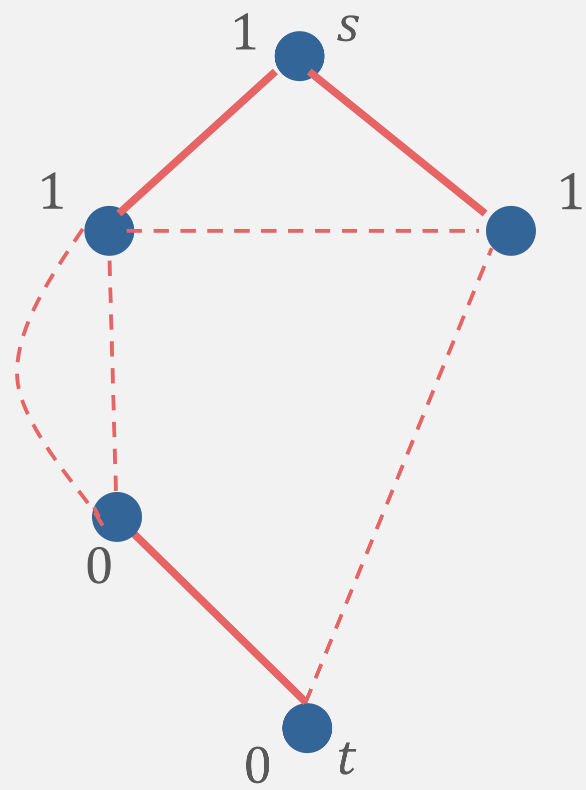


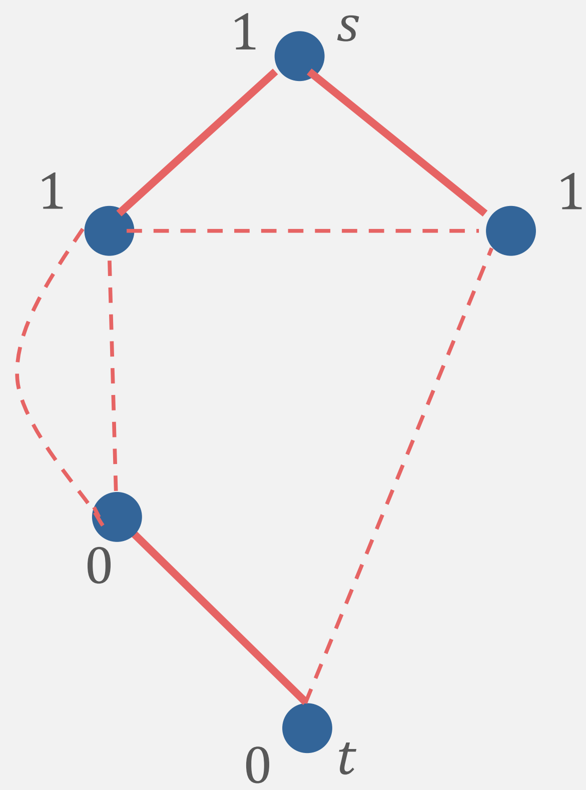
Effective Capacitance

Cut energy:

$$\sum_{edges} (Potential\ Energy\ Difference)^2$$

Effective Capacitance: $C_{s,t}(G')$
Smallest cut energy of any valid potential energy between s to t on G'.

Graph G':



Algorithm Performance:

st-connectivity algorithm complexity =

$$O\left(\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)}\ \sqrt{\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G')}\right)$$

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[Belovs, Reichard, '12]

[JJ**K**P, in progress]

Example

What is quantum complexity of deciding $AND(x_1, x_2, \ldots, x_N)$, promised

- All $x_i = 1$, or
- At least $\sqrt{N}$ input variables are 0.

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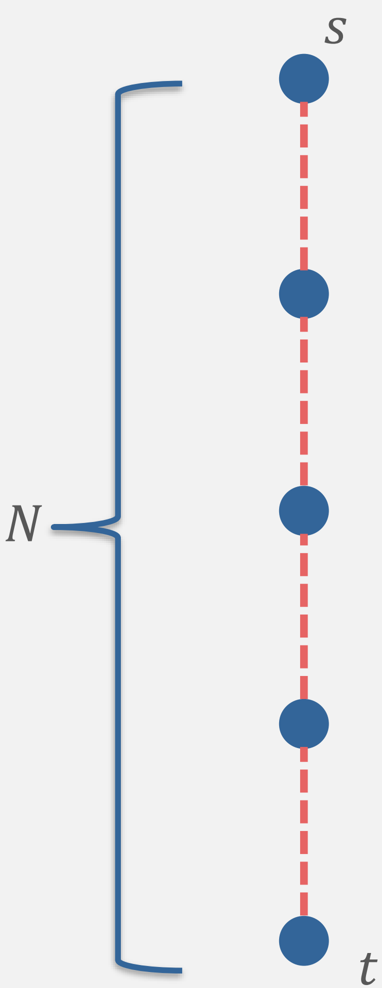

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$\Updownarrow$

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- s and t are connected, or
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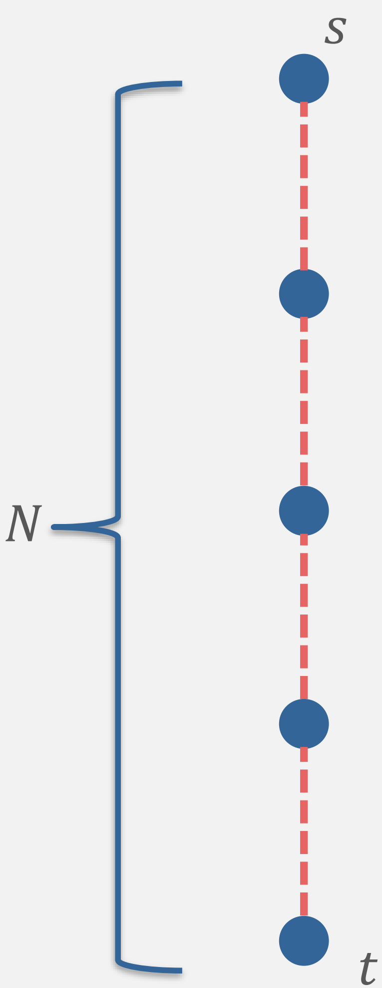

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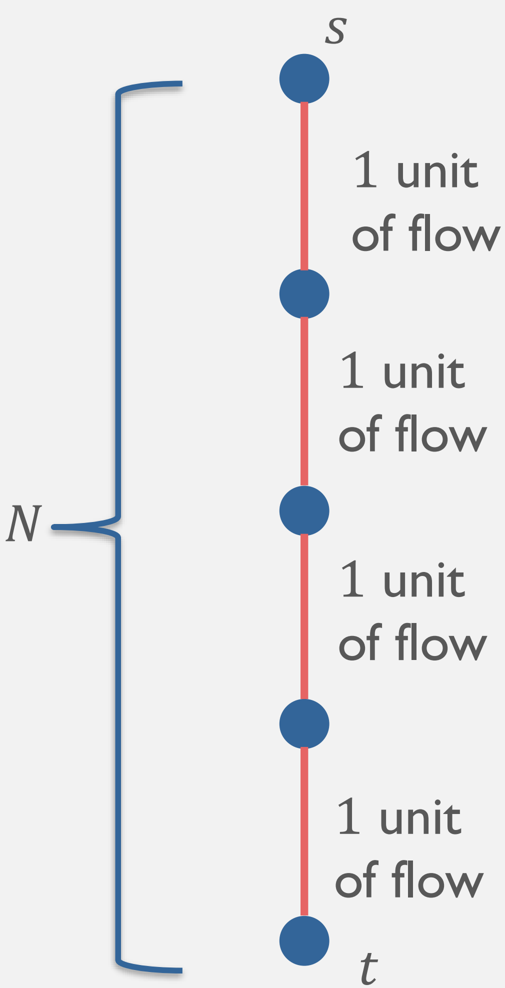

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$$\max_{G\in\mathcal{H}:connected} R_{s,t}(G) = N$$

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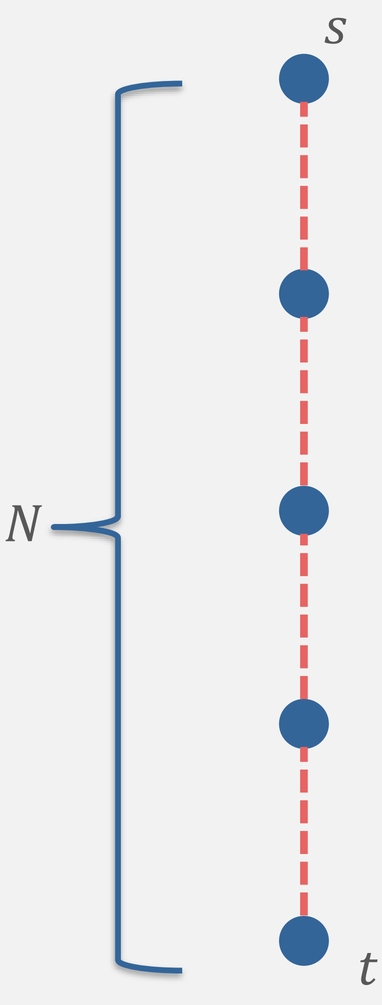

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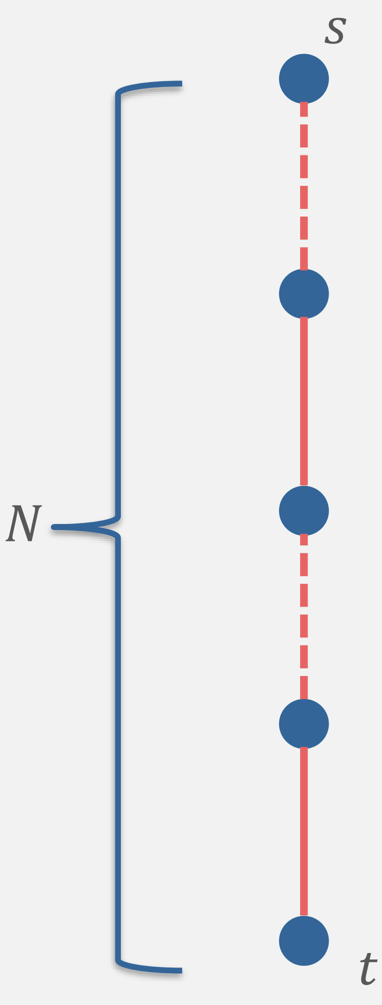

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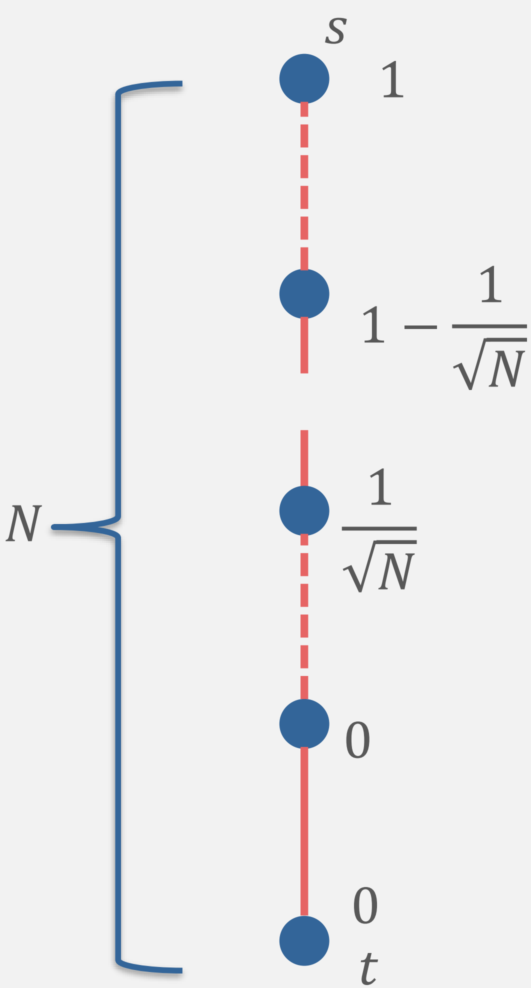

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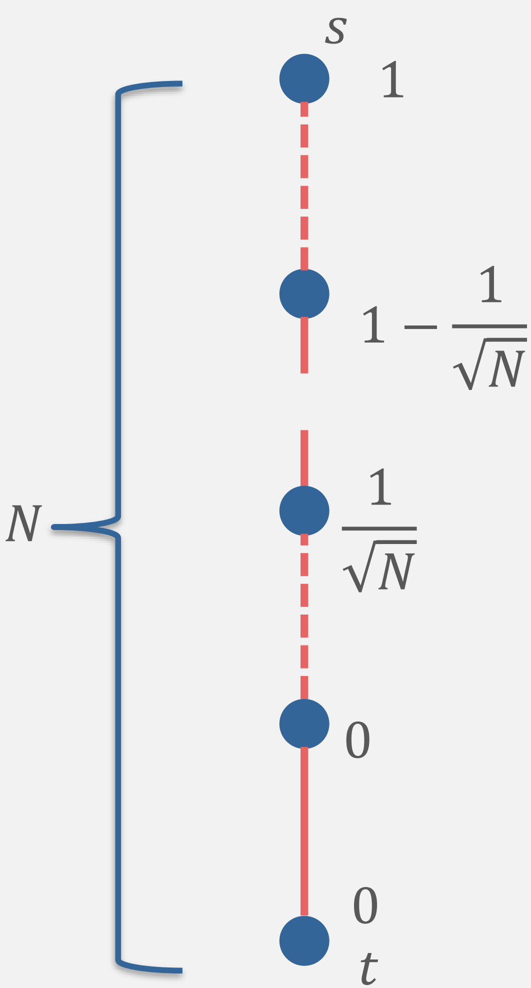

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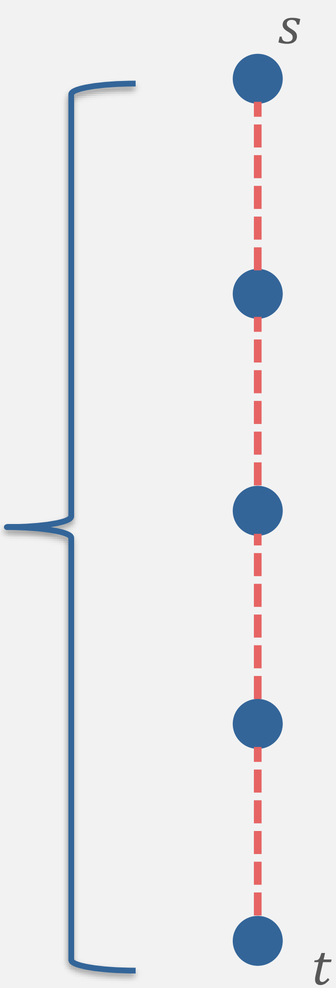
$$\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G') = \sqrt{N}\times\left(\frac{1}{\sqrt{N}}\right)^2 = \frac{1}{\sqrt{N}}$$

Example



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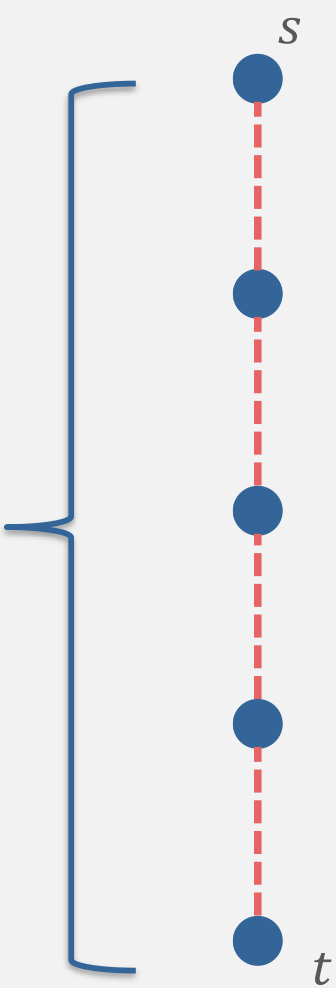
$$\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)} \qquad \sqrt{\max_{G\in\mathcal{H}:not\ connected} R_{s',t'}(G')}$$

$$N \qquad\qquad 1/\sqrt{N}$$

Quantum complexity is $O\left(N^{1/4}\right)$

Example



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$$\downarrow \qquad\qquad \downarrow$$

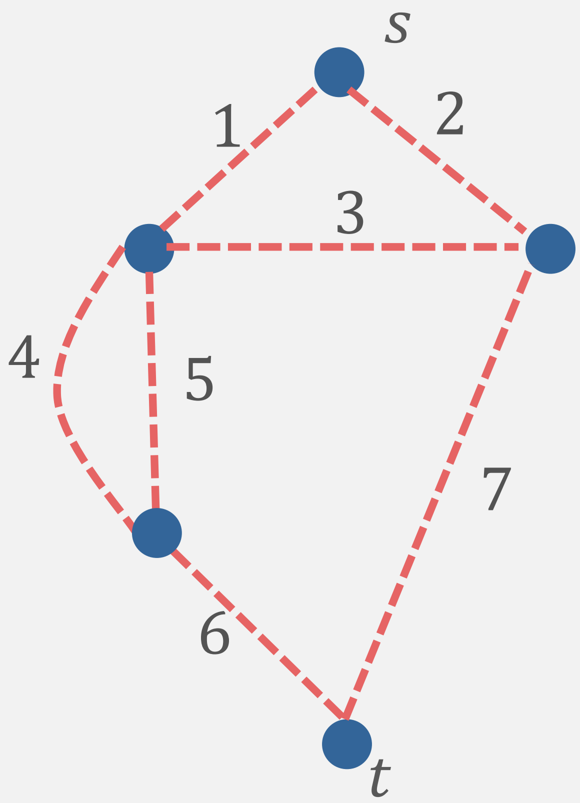
$$N \qquad\qquad 1/\sqrt{N}$$

Quantum complexity is $O\left(N^{1/4}\right)$

Randomized classical complexity is $\Omega\left(N^{1/2}\right)$

New Example

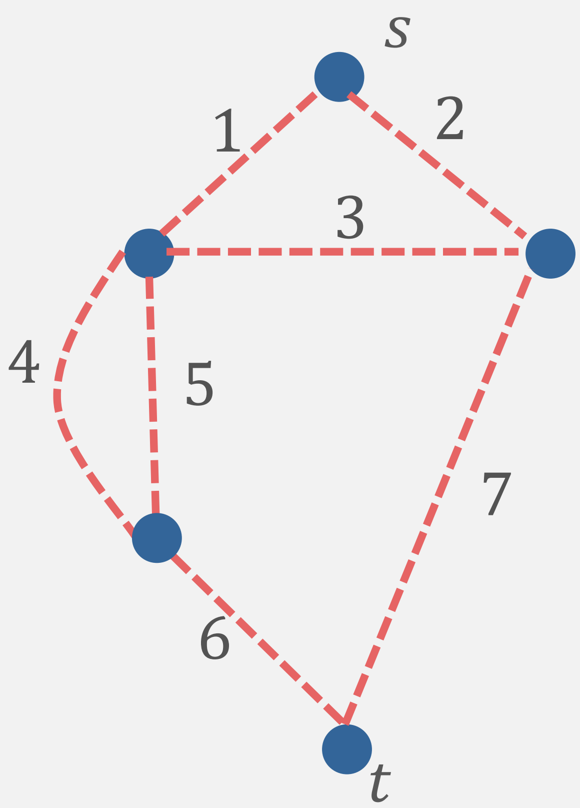
Connectivity – is every vertex connected to every other vertex?



New Example

Connectivity – is every vertex connected to every other vertex?

Connectivity=
$(st - conn) \wedge (su - conn) \wedge (uv - conn) \ldots$



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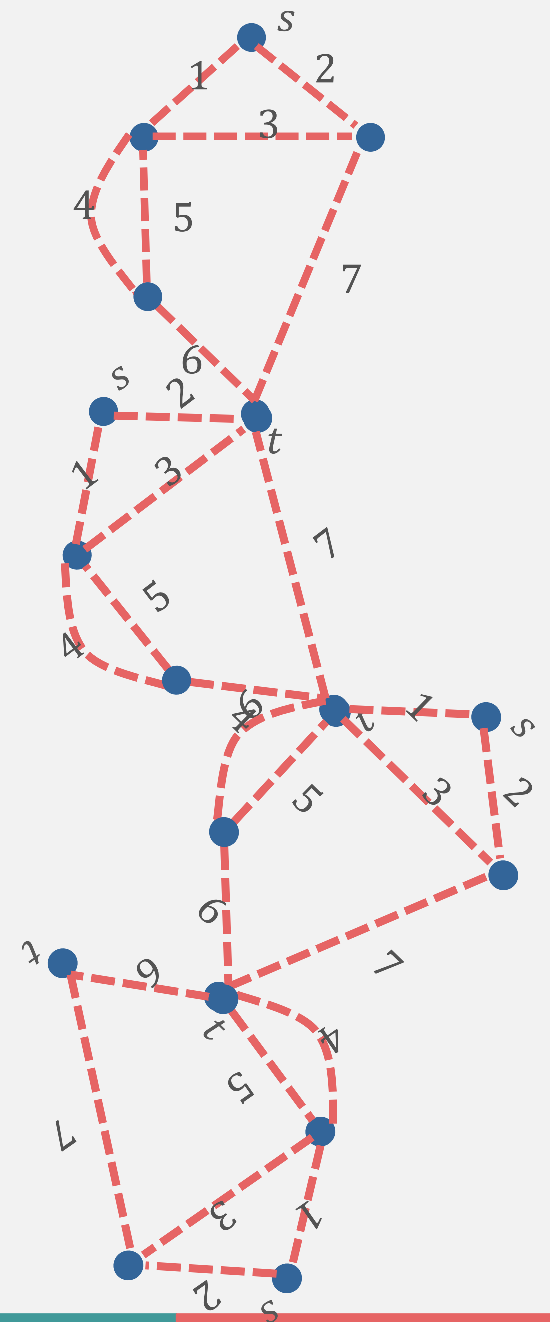
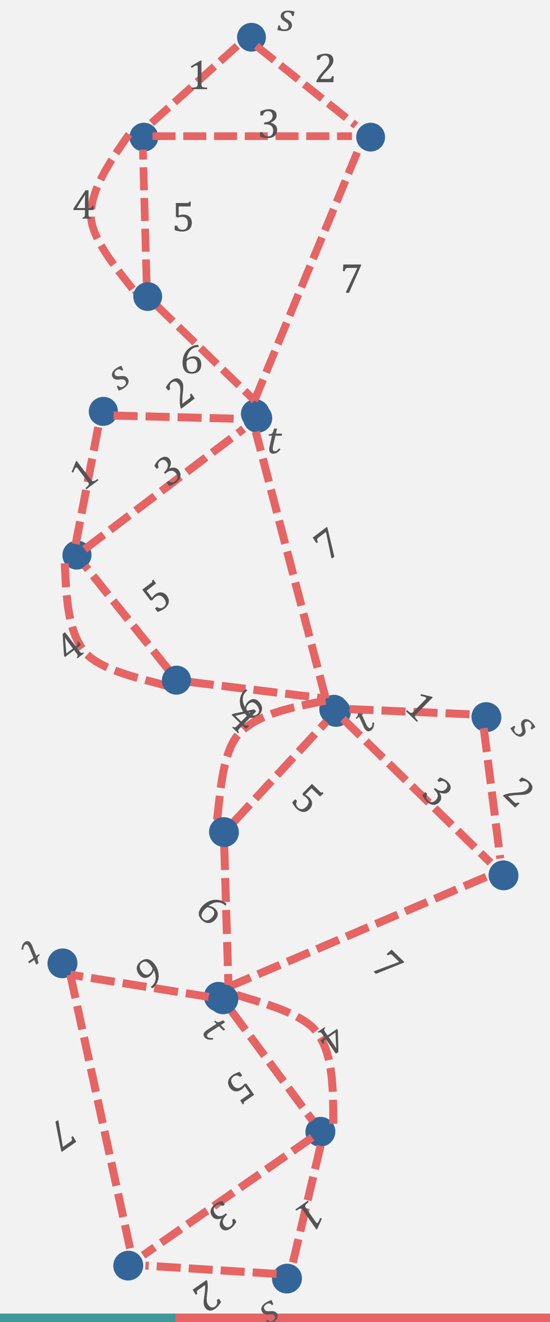


New Example

Connectivity – is every vertex connected to every other vertex?

Results:

- Worst case: $O(n^{3/2})$ (n = # vertices)
- Promised
  - YES – diameter is D
  - NO – every connected component has at most n^* vertices
  - $O\left(\sqrt{nn^*D}\right)$



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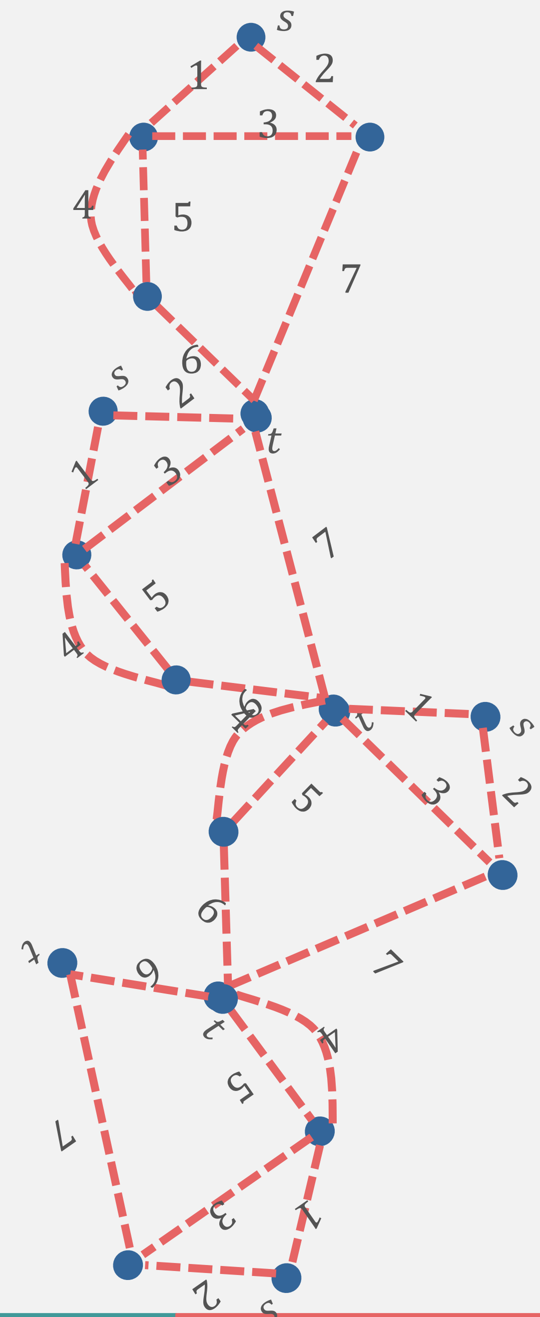

Results:

- Worst case: $O(n^{3/2})$ (n = # vertices)
- Promised
  - YES – diameter is D
  - NO – every connected component has at most K vertices
  - $O\left(\sqrt{nKD}\right)$

(Diameter result previously discovered by Arins using slightly different approach)

The Algorithm

Span Program

- Span vectors
- Target vector

The input to the problem determines which subset of span vectors are allowed.

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If target vector is in span of the allowed span vectors, then function evaluates to 1 on that input. Otherwise, evaluates to 0.

Thus span program encodes a function.

The Algorithm

Span Program

- Span vectors
- Target vector

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Infinite number of span programs can encode the same function

Given a span program, can create a quantum algorithm to evaluate the corresponding function (create a quantum walk whose dispersion operators are based on the vectors)

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The efficiency of the span program is a (relatively) simple function of the vectors.

There is always a span program algorithm that is optimal (and many that are not optimal.)

Open Questions and Current Directions

- When is our algorithm optimal for Boolean formulas? (Especially partial/read-many formulas)
- Are there other problems that reduce to st-connectivity? (Perhaps all?)
- What is the classical time/query complexity of st-connectivity in the black box model? Under the promise of small capacitance/resistance?
- Does our reduction from formulas to connectivity give good classical algorithms too?
- How to choose weights?