What does the effective resistance of electrical circuits have to do with quantum algorithms?

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(Simplest) Answer

st – connectivity: is there a path from $s$ to $t$?
(Simplest) Answer

*st – connectivity:* is there a path from s to t?
(Simplest) Answer

We can turn this into a circuit by attaching leads to $s$ and $t$, and putting $1 \, \Omega$ resistors wherever edges exist.
(Simplest) Answer

Speed of quantum algorithm for st-connectivity depends on effective resistance of this circuit! (Lower effective resistance -> quicker detection of path)

[Belovs, Reichardt ‘12]
Applications of st-Connectivity

• Important (social) network problem
• Problem is a useful subroutine for many problems
  – Is there a length-k path? [Belovs, Reichardt ‘12]
  – Is a graph a forest? [Cade, Montanaro, Belovs ‘16]
  – Is a graph bipartite? [Cade, Montanaro, Belovs ‘16]
Applications of st-Connectivity

- Important (social) network problem
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  - Is there a length-k path?
  - Is a graph a forest?
  - Is a graph bipartite?
  - Boolean formula evaluation

NEW
Our results:

Improved analysis of quantum algorithm for st-connectivity (with even more effective resistance than before!)

Use this algorithm to get improved quantum algorithm for Boolean formula evaluation
Outline

• Previous algorithm for st-connectivity
• Improved analysis for planar graphs
• Application to Boolean formulas
Black Box Algorithm

- $e_i = 1$ if $i^{th}$ edge is there
- $e_i = 0$ if edge is not there
$R(G)$ is the effective resistance of the circuit created by attaching a voltage between $s$ and $t$, and 1 Ω resistors at all edges.
Previous Quantum Algorithm

st-connectivity algorithm time/queries ~

\[ \sqrt{\max_{G: \text{connected}} R(G)} \sqrt{\max_{G: \text{not connected}} |G|} \]

\# of edges in graph \( G \)

[Belovs, Reichardt '12]
Planar Graph

Planar

Not Planar
Planar Graph Dual(ish)
Planar Graph Dual(ish)
Planar Graph Dual(ish)

- If an edge is not present in $G$, it is present in $G'$
Planar Graph Dual(ish)

- If there is an $st$-path, there is no $s't'$-path.
- If there is an $s't'$-path, there is no $st$-path.
Planar Graph Dual(ish)
Planar Graph Dual(ish)

\( R(G') \) is the effective resistance of the circuit created by attaching a voltage between \( s' \) and \( t' \), and 1 \( \Omega \) resistors at all edges.
Improved Quantum Algorithm for st-connectivity

Planar graph† st-connectivity algorithm time/queries =

\[ \sqrt{\max_{G: \text{connected}} R(G)} \sqrt{\max_{G: \text{not connected}} R(G')} } \]

† with \( s, t \) on external face
Application to Boolean Formulas

- **AND**: outputs 1 if all inputs are 1
- **OR**: outputs 1 if any input is 1
- **Value**: 0 or 1

The diagram represents a function $f(x)$ with inputs $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$, $x_8$, and $x_9$. The function is constructed using AND and OR gates.
Application to Boolean Formulas

\( \land \): outputs 1 if all inputs are 1

\( s \) and \( t \) are connected if all subgraphs are connected
Application to Boolean Formulas

\[ \bigwedge \quad \text{AND: outputs 1 if all inputs are 1} \]

\[ \bigvee \quad \text{OR: outputs 1 if any input is 1} \]

\[ s \text{ and } t \text{ are connected if all subgraphs are connected} \]

\[ s \text{ and } t \text{ are connected if any subgraph is connected} \]
Application to Boolean Formulas

\[ f(x) = \bigwedge \left( \bigvee \left( \bigwedge \left( \bigvee \left( \bigwedge \left( \bigvee \left( \bigwedge x_1, x_2, x_3, x_4 \right) \right) \right) \right) \right) \]
Application to Boolean Formulas

\[ f(x) = x_1 \land (x_2 \lor x_3 \land (x_4 \lor x_5 \land (x_6 \lor (x_7 \land (x_8 \land x_9)))) \]
Application to Boolean Formulas

\[ f(x) = x_1 \land (x_2 \lor x_3 \land x_4) \land x_{10} \]
If we put edges where $x_i = 1$, $s$ and $t$ are connected iff $f(x) = 1$!
Application to Boolean Formulas

- The graph associated with a formula will always be planar, with $s, t$ on external face.
- Can use our st-connectivity algorithm! Time required depends on the effective resistance of circuit corresponding to corresponding graph.
Open Questions

• When is our algorithm optimal for Boolean formulas?
• Can we extend these ideas to non-planar graphs?
• Are there other problems that reduce to st-connectivity?
• What is the classical time/query complexity of st-connectivity? Can we relate it to effective resistance?