Robust Single-Qubit Process Calibration via Robust Phase Estimation

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Imagine...
Imagine...
Imagine...

- All gates need to be tuned
- State preparation is off
- Measurements are off

Want to quickly determine imperfections in gate controls and then tune to fix.
Need to Calibrate Operations

Amplitude of Rotation: $\pi$

Axis of Rotation: $\hat{x}$

Ideal Unitary $U$
Need to Calibrate Operations

Amplitude of Rotation:
\[ A = \pi (1 + \epsilon) \]

Axis of Rotation:
\[ \cos(\phi)\hat{x} + \sin(\phi)\hat{z} \]

Unitary \( \tilde{U} \)
Need to Calibrate Operations

Amplitude of Rotation:
\[ A = \pi (1 + \varepsilon) \]

\( \varepsilon \) is an “Amplitude Error”

Axis of Rotation:
\[ \cos(\phi)|x\rangle + \sin(\phi)|z\rangle \]

\( \phi \) is an “Off-Resonance Error”
How to Estimate Control Errors

Ad hoc Rabi – Ramsey Sequences.
How to Estimate Control Errors

Process Tomography [Chuang & Nielsen ’97]
How to Estimate Control Errors

Process Tomography

- Need perfect state preparation and measurement
- Need perfect additional gates
- Time consuming: need to learn 12 parameters to extract $\phi$ and $\epsilon$
How to Estimate Control Errors

Process Tomography

Randomized Benchmarking Tomography
How to Estimate Control Errors

Randomized Benchmarking

- Don’t need perfect state preparation and measurement
- Must be able to perform single-qubit Cliffords (although not perfectly)
- Time consuming: need to learn 9 parameters to extract $\phi$ and $\epsilon$
How to Estimate Control Errors

Process Tomography
Randomized Benchmarking
How to Estimate Control Errors

GST
[Blume-Kohout et al ‘13]

Process Tomography

Randomized Benchmarking
How to Estimate Control Errors

\[ \vec{U} \]

- Process Tomography
- Randomized Benchmarking
- GST
How to Estimate Control Errors

Robust Phase Estimation

- Don’t need perfect state preparation and measurement
- No additional gates, except Z-rotations
- Learn $\phi$ and $\epsilon$ with optimal efficiency
- Non-adaptive
- Accommodates additional errors like depolarizing noise.
Outline

• Motivation for Robust Phase Estimation
• Robust phase estimation
• Application to Parameter Estimation
Phase Estimation [Higgins et al. ‘09]

Can sample from 2 binomial random variables with probability of “heads”

\[
\frac{1 + \sin k\theta}{2}, \quad \frac{1 + \cos k\theta}{2}
\]

For \( k \) in \( \mathbb{Z} \), each in time \( k \)
Phase Estimation [Higgins et al. ‘09]

Can sample from 2 binomial random variables with probability of “heads”

$$\frac{1 + \sin k\theta}{2}, \quad \frac{1 + \cos k\theta}{2}$$

For $k$ in $\mathbb{Z}$, each in time $k$

$k = 1$

Can estimate $\theta$ with standard deviation $\sigma(\theta) \sim \frac{1}{\sqrt{T}}$
Phase Estimation [Higgins et al. ‘09]

Can sample from 2 binomial random variables with probability of “heads”

\[
\begin{align*}
&\frac{1 + \sin k\theta}{2}, \\
&\frac{1 + \cos k\theta}{2}
\end{align*}
\]

For \( k \) in \( \mathbb{Z} \), each in time \( k \)

\( k = 1 \)
Phase Estimation [Higgins et al. ‘09]

Can sample from 2 binomial random variables with probability of “heads”

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\frac{1 + \sin k\theta}{2}, \quad \frac{1 + \cos k\theta}{2}
\]

For \( k \) in \( \mathbb{Z} \), each in time \( k \)

\( k = 1 \quad k = 2 \)
Phase Estimation [Higgins et al. ‘09]

Can sample from 2 binomial random variables with probability of “heads”

\[
\begin{align*}
\frac{1 + \sin k\theta}{2}, & \quad \frac{1 + \cos k\theta}{2}
\end{align*}
\]

For \( k \) in \( \mathbb{Z} \), each in time \( k \)

\[ k = 1 \quad k = 2 \quad k = 4 \]
Phase Estimation [Higgins et al. ‘09]

Can estimate $\theta$ with standard deviation $\sigma(\theta) \sim \frac{1}{T}$

Optimal – by information theory.

Can sample from 2 binomial random variables with probability of “heads”

$$\frac{1 + \sin k\theta}{2}, \frac{1 + \cos k\theta}{2}$$

For $k$ in $\mathbb{Z}$, each in time $k$

$k = 1 \quad k = 2 \quad k = 4 \quad k = 8$
Robust Phase Estimation

Can sample from 2 binomial random variables with probability of “heads”

$$\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}$$

For $k$ in $\mathbb{Z}$, each in time $k$
Robust Phase Estimation

Can sample from 2 binomial random variables with probability of “heads”

\[
\frac{1 + \sin \theta}{2} + \delta_{k_1}, \quad \frac{1 + \cos \theta}{2} + \delta_{k_2}
\]

Using only \( k = 1 \) can’t get an accurate estimate!

\( \theta \)
Robust Phase Estimation

Can sample from 2 binomial random variables with probability of "heads"

\[
\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
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For \( k \) in \( \mathbb{Z} \), each in time \( k \)

\[ k = 1 \]
Robust Phase Estimation

Can sample from 2 binomial random variables with probability of “heads”

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\frac{1 + \sin k\theta}{2} + \delta_{k1}, \\
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For \( k \) in \( \mathbb{Z} \), each in time \( k \)

\( k = 1 \quad k = 2 \)
Robust Phase Estimation

Can sample from 2 binomial random variables with probability of “heads”

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\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
\]

For \(k\) in \(\mathbb{Z}\), each in time \(k\)

\[k = 1 \quad k = 2 \quad k = 4\]

Can estimate \(\theta\) with standard deviation \(\sigma(\theta) \sim \frac{1}{T}\), as long as \(|\delta_k| < .35\) for all \(k\).
Robust Phase Estimation

Can estimate $\theta$ with standard deviation $\sigma(\theta) \sim \frac{1}{T}$, as long as $|\delta_k| < .35$ for all $k$.

...but need upper bound on size of $\delta$ to know how many extra samples to take.

Can sample from 2 binomial random variables with probability of “heads”

$$\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}$$

For $k$ in $\mathbb{Z}$, each in time $k$

$k = 1 \quad k = 2 \quad k = 4$
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities like:

\[
\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
\]
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
\]

\[
|\langle 0|\tilde{U}^k|0\rangle|^2 = \frac{1 + \cos kA}{2} + \sin^2 \frac{kA}{2} \sin^2 \phi
\]

\[
|\langle 0|\tilde{U}^k|\rightarrow\rangle|^2 = \frac{1 + \sin kA}{2} - \sin kA \sin^2 \frac{\phi}{2}
\]

\(A = \pi (1 + \epsilon)\) is total amplitude of rotation
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
|\langle 0 | \widetilde{U}^k | 0 \rangle|^2 = \frac{1 + \cos kA}{2} + \frac{1 + \sin kA}{2} - \sin kA \sin^2 \phi \frac{\phi}{2}
\]

\[A = \pi(1 + \epsilon)\text{ is total amplitude of rotation}\]
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
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|\langle 0|\tilde{U}^k|0 \rangle|^2 = \frac{1 + \cos kA}{2} + \sin^2 \frac{kA}{2} \sin^2 \phi
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|\langle 0|\tilde{U}^k \rightarrow \rangle|^2 = \frac{1 + \sin kA}{2} - \sin kA \sin^2 \frac{\phi}{2}
\]

Heisenberg limited! Estimate of \( \epsilon \) with standard deviation \( \sigma(\epsilon) \sim \frac{1}{N'} \)

where \( N \) is the number of times \( \tilde{U} \) is applied.
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
\begin{align*}
\frac{1 + \sin k\theta}{2} + \delta_{k1}, & \quad \frac{1 + \cos k\theta}{2} + \delta_{k2} \\
\end{align*}
\]

\[
\begin{align*}
|\langle 0 | (Z_{-\pi/2} \tilde{U} Z_{\pi/2} \tilde{U} Z_{-\pi/2})^k |0 \rangle|^2 & \approx \frac{1 + \cos m_\epsilon k\phi}{2} + O(\epsilon^2) \\
|\langle 0 | (Z_{-\pi/2} \tilde{U} Z_{\pi/2} \tilde{U} Z_{-\pi/2})^k |\rightarrow \rangle|^2 & \approx \frac{1 + \sin m_\epsilon k\phi}{2} + O(\epsilon^2)
\end{align*}
\]

Heisenberg limited! Estimate of $\phi$ with standard deviation $\sigma(\phi) \sim \frac{1}{N'}$, where $N$ is the number of times $\tilde{U}$ is applied.
Additional Errors

Looks like need perfect $|\langle 0 | \widetilde{U}^k | \rightarrow \rangle|^2$
Additional Errors

Looks like need perfect $\left| \langle 0 | \tilde{U}^k | \rightarrow \rangle \right|^2$

All of the following errors simply contribute to $\delta$ errors

• Imperfect state preparation
• Imperfect measurement
• Additional errors like depolarizing errors
• Imperfect Z rotation
Example: State Preparation Errors Add to $\delta$ errors

Want: $|\langle 0 | \tilde{U}^k | \rightarrow \rangle|^2$

Suppose can only prepare $|0\rangle$, measure $|0\rangle$. No perfect X-rotation, so can’t prepare $|\rightarrow\rangle$. Instead prepare $\rho'_{\rightarrow}$.
Example: State Preparation Errors Add to $\delta$ errors

Want: $|\langle 0 | \tilde{U}^k | \rightarrow \rangle|^2$

Suppose can only prepare $|0\rangle$, measure $|0\rangle$. No perfect X-rotation, so can’t prepare $|\rightarrow\rangle$. Instead prepare $\rho' \rightarrow$

Trace Distance: $D(\rho, \sigma) = \text{maximum difference in probability between any two experiments on states } \rho, \sigma$.

Thus if use $\rho \rightarrow$ instead of $|\rightarrow\rangle$, $\delta$ error changes by at most $D(\rho' \rightarrow, |\rightarrow\rangle \langle \rightarrow |)$
Example: State Preparation Errors Add to $\delta$ errors

Want experiment with outcome probability:

$$|\langle 0 | \tilde{U}^k | \rightarrow \rangle|^2 = \text{tr}(M_0 \tilde{U}^k (\rho_{\rightarrow}))$$

Have experiment with outcome probability:

$$\text{tr}(M_0 \tilde{U}^k (\rho'_{\rightarrow}))$$

$$\text{tr}(M_0 \tilde{U}^k (\rho'_{\rightarrow})) = \text{tr}(M_0 \tilde{U}^k (\rho_{\rightarrow})) - \text{tr}(M_0 \tilde{U}^k (\rho'_{\rightarrow} - \rho_{\rightarrow}))$$

Have Want $< D(\rho_{\rightarrow}, \rho'_{\rightarrow})$
Additional Errors

\[ \left| \langle 0 | \tilde{U}^k | \rightarrow \rangle \right|^2 \]

All of the following errors simply contribute to \( \delta \) errors

- Imperfect state preparation
- Imperfect measurement
- Additional errors like depolarizing errors
- Imperfect Z rotation

Can incorporate any error, as long as can bound how much that error will shift your outcome probability (and total shift is less than .35)
Bounding $\delta$ Errors

Need upper bounds on
- Size of $\phi, \epsilon$
- Trace distance between ideal and true state preparation
- “Trace distance” between ideal and true measurement

We provide simple (length-0/1) sequences to upper bound these quantities.
Sample Procedure

1. Bound $\delta$ errors
2. Choose # of samples to take each round based on size of $\delta$ errors and desired precision
3. Robust phase estimation on $\epsilon$.
4. Robust phase estimation on $\theta$.
5. Use controls to correct errors, repeat.
Sample Procedure

1. Bound $\delta$ errors
2. Choose # of samples to take each round based on size of $\delta$ errors and desired precision
3. Robust phase estimation on $\epsilon$.
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5. Use controls to correct errors, repeat.

If bad $\delta$ bounds, $\epsilon$, $\theta$ estimates accurate, just not as precise.
Sample Procedure

1. Bound $\delta$ errors
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3. Robust phase estimation on $\epsilon$.
4. Robust phase estimation on $\theta$.
5. Use controls to correct errors, repeat.

If bad $\delta$ bounds, $\epsilon,\theta$ estimates accurate, just not as precise.

e.g. Controls have 5 digits of precision. Estimate $\epsilon,\theta$ to 5 digits of precision, but after correcting still inaccurate at 3 digits of precision. $\delta$ errors could be cause.
Recap:

Robust Phase Estimation

- Don’t need perfect state preparation and measurement
- No additional gates, except Z-rotations
- Learn $\phi$ and $\epsilon$ with optimal efficiency
- Non-adaptive
- Accommodates additional errors like depolarizing noise.

Open Questions

- Multi-Qubit Operations?
- No perfect Z-rotation?
- Connection to Randomized Benchmarking
Think this might be useful?

Arxiv: 1502.02677
Imagine…

[Monroe Lab]

$^{171}\text{Yb}^+$

2 $\mu$m
Imagine...

- All gates need to be tuned
- State preparation is off
- Measurements are off

Want to quickly determine imperfections in gate controls and then tune to fix.
Proof Sketch
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
\frac{1 + \sin k\theta}{2} + \delta_{k1}, \quad \frac{1 + \cos k\theta}{2} + \delta_{k2}
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\[
|\langle 0|\widetilde{U}^k|0\rangle|^2 = \frac{1 + \cos kA}{2} + \frac{\sin^2 kA}{2} \sin^2 \phi
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\[
|\langle 0|\widetilde{U}^k|\rightarrow\rangle|^2 = \frac{1 + \sin kA}{2} - \sin kA \sin^2 \frac{\phi}{2}
\]

Don’t have perfect state prep and measurement? OK! Just add to \(\delta\) error.
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

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|\langle 0|\widetilde{U}^k|0\rangle|^2 = \frac{1 + \cos k\theta}{2} + \delta_{k1}, \quad \frac{1 + \sin k\theta}{2} + \delta_{k2}
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|\langle 0|\widetilde{U}^k|\rightarrow\rangle|^2 = \frac{1 + \cos kA}{2} + \sin^2 \frac{kA}{2} \sin^2 \phi
\]

\[
\frac{1 + \sin kA}{2} - \sin kA \sin^2 \frac{\phi}{2}
\]

Have depolarizing errors? OK! Just add to $\delta$ errors.
Robust Phase Estimation for Gate Estimation

Want 2-outcome experiments with probabilities:

\[
\begin{align*}
|\langle 0|\tilde{U}^k|0\rangle|^2 &= \frac{1 + \cos kA}{2} + \delta_{k1}, \\
|\langle 0|\tilde{U}^k|\rightarrow\rangle|^2 &= \frac{1 + \sin kA}{2} - \sin kA \sin^2 \frac{\phi}{2} + \delta_{k2}
\end{align*}
\]

Heisenberg limited! Estimate of \( \epsilon \) with standard deviation \( \sigma(\epsilon) \sim \frac{1}{N} \), where \( N \) is the number of times \( \tilde{U} \) is applied.