Phase Retrieval Using Unitary 2-Designs

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Familiar Problem: Phase Retrieval

Phase Retrieval:

Learn unknown signal $x \in \mathbb{C}^d$, given noisy quadratic measurements:

$$y_i = |a_i^* x|^2 + \epsilon_i$$

Where $a_i \in \mathbb{C}^d$ are chosen by observer, ϵ_i are unknown noise, using as few measurement settings as possible.

Variant: Phase Retrieval using Unitaries

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Phase Retrieval using Unitaries:

Learn unknown unitary matrix $U \in \mathbb{C}^{d \times d}$, given noisy quadratic measurements:

$$y_i = |Tr(C_i^* U)\ |^2 + \epsilon_i$$

Where $C_i \in \mathbb{C}^{d \times d}$ are unitary matrices chosen by observer, ϵ_i are unknown noise, using as few measurement settings as possible.

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N.B.:

- $|Tr(C_\mathrm{i}^* U)\ |^2 = |\mathrm{vec}(C_i)^* \mathrm{vec}(U)|^2$
- C_i^* is conjugate transpose of C_i

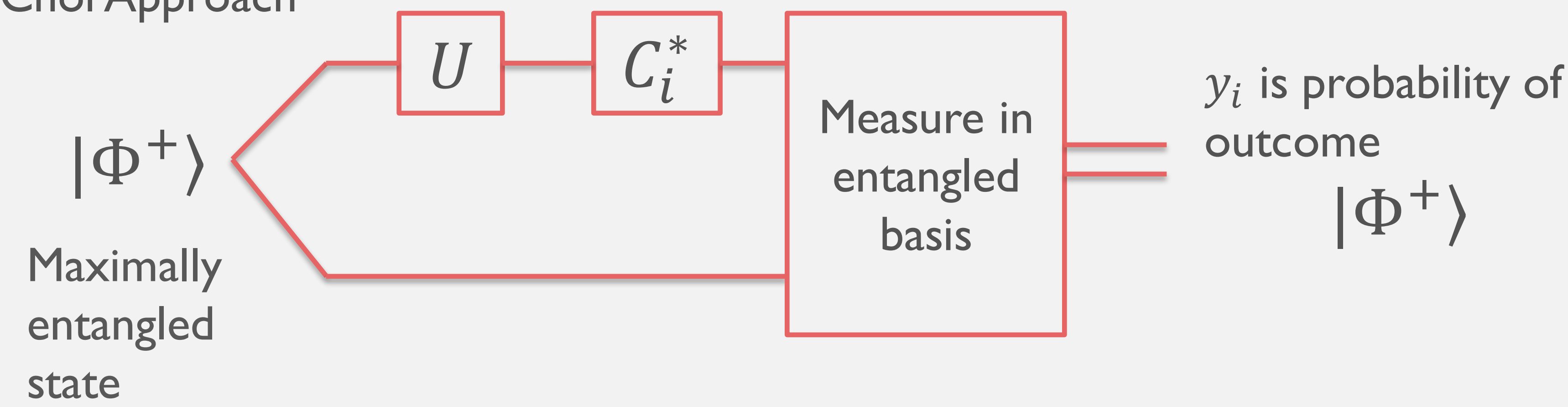
Why Phase Retrieval Using Unitaries?

- Unitaries are basic building blocks of a quantum computer
- Physical implementations often not correct – need to find errors.

How Phase Retrieval Using Unitaries?

Measurement Schemes to obtain $y_i = |Tr(C_i^* U)|^2 + \epsilon_i$

1. Choi Approach

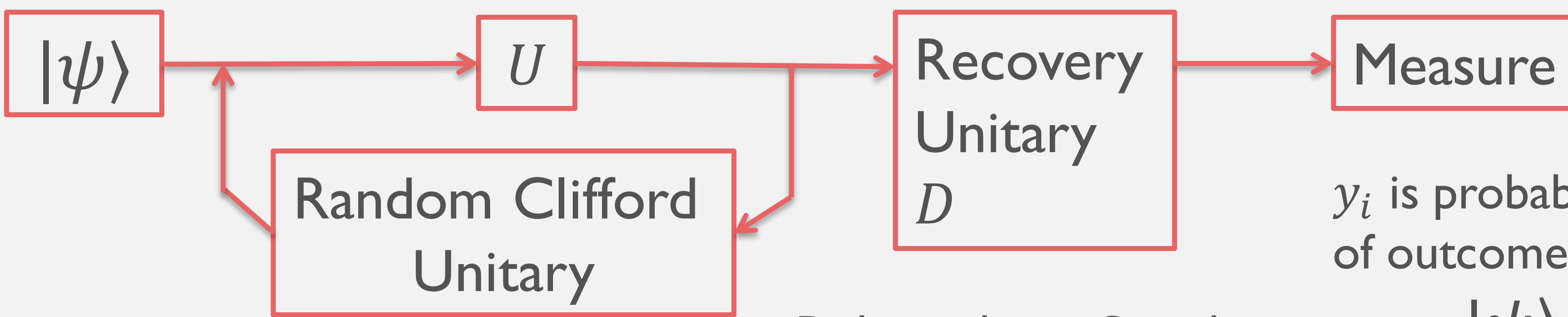


- Difficult to prepare entangled state and measure in entangled basis
- Can't characterize unitaries acting on full system.

How Phase Retrieval Using Unitaries?

Measurement Schemes to obtain $y_i = |Tr(C_i^* U)|^2 + \epsilon_i$

2. Randomized Benchmarking



D depends on C_i and sequence of random unitaries

y_i is probability of outcome $|\psi\rangle$

- Good: inherently protected from SPAM errors, no entanglement needed
- Bad: C_i must be a Clifford Unitary.

Phase retrieval possible when C_i chosen from Cliffords?

Our Results:

How does our choice of C_i affect our ability to learn U?

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2. Phase retrieval works pretty well when C_i chosen from a unitary 2-design.

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   Cliffords form a unitary 3-design! [Zhu; Webb; Kueng and Gross, 2015]

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Unitary t-design is a set of unitaries that has same t^{th} order moments as the Haar distribution

2. Phase ret … om a
   unitary 2-

Our Results:

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   - Use PhaseLift algorithm
   - Matrix analog of vector phase retrieval result using vector 4-designs [Kueng, Rauhut, and Terstiege, 2014]

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2. Phase retrieval works pretty well when C_i chosen from a unitary 2-design.
   - Note: no-go result for PhaseLift using vector 2-designs [Gross et al, 2013]
   - PhaseLift is approximately correct, for most unitaries

More Details

- PhaseLift: Lifts vector to matrix, solve convex optimization problem on larger space
- Our case: lifts matrix to larger matrix, solve convex optimization problem on larger space

$$U \to vec(U)vec(U)^* \in \mathbb{C}^{d^2 \times d^2}$$

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  - ✓ For unitary 4-design, can bound using properties of 4^{th} moment of Haar random unitaries (using Weingarten functions and commutative diagrams)

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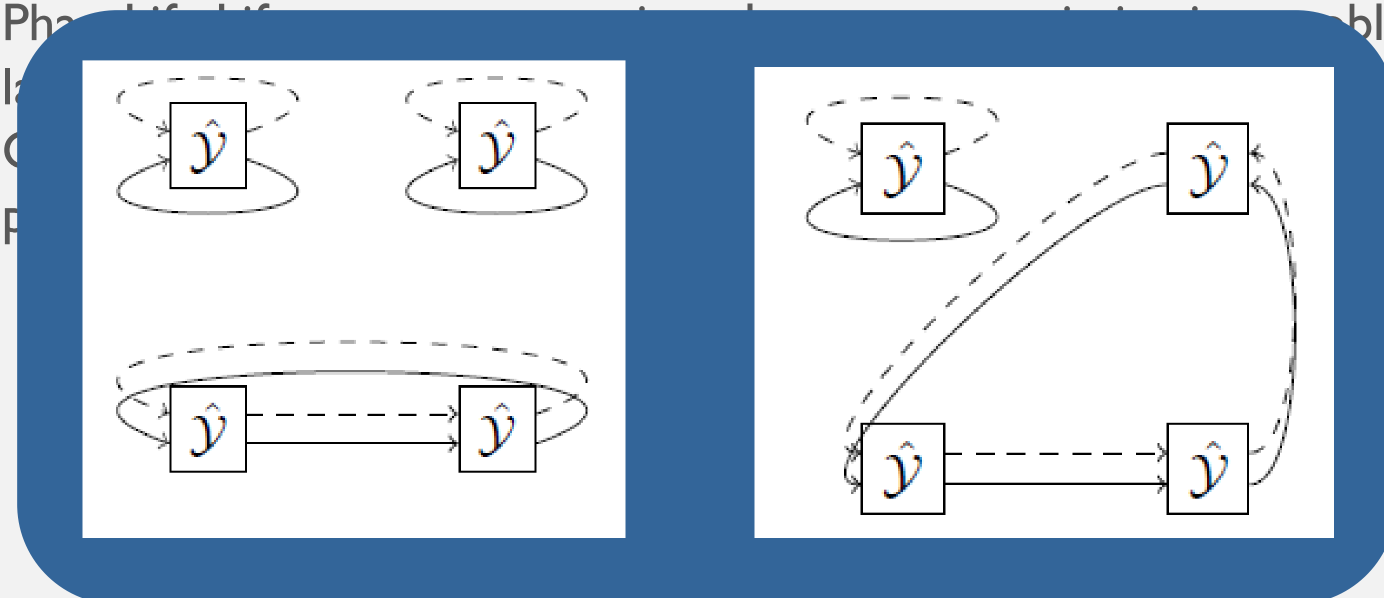

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- Follow strategy similar to [Keung et al. 2014] for state 4-designs.
- Key component: bounding expectation of 4th power of certain term
  - ✓ For unitary 4-design, can bound using properties of 4^{th} moment of Haar random unitaries
  - ✓ For unitary 2-design, can bound using properties of 2^{nd} moment of Haar random unitaries, AND non-spikiness condition.

More Details

Non-spikiness condition:

Let $\tilde{G}$ be a finite set of unitary matrices in $\mathbb{C}^{d\times d}$. We say that a unitary matrix $U \in \mathbb{C}^{d\times d}$ is *non-spiky* with respect to $\tilde{G}$ (with parameter $\beta \geq 0$) if the following holds:

$$|\mathrm{tr}(C^\dagger U)|^2 \leq \beta, \quad \forall C \in \tilde{G}. \tag{I.13}$$

Fact: Almost all unitary matrices are non-spiky when $\beta \sim \log|\tilde{G}|$!

What about Cliffords?

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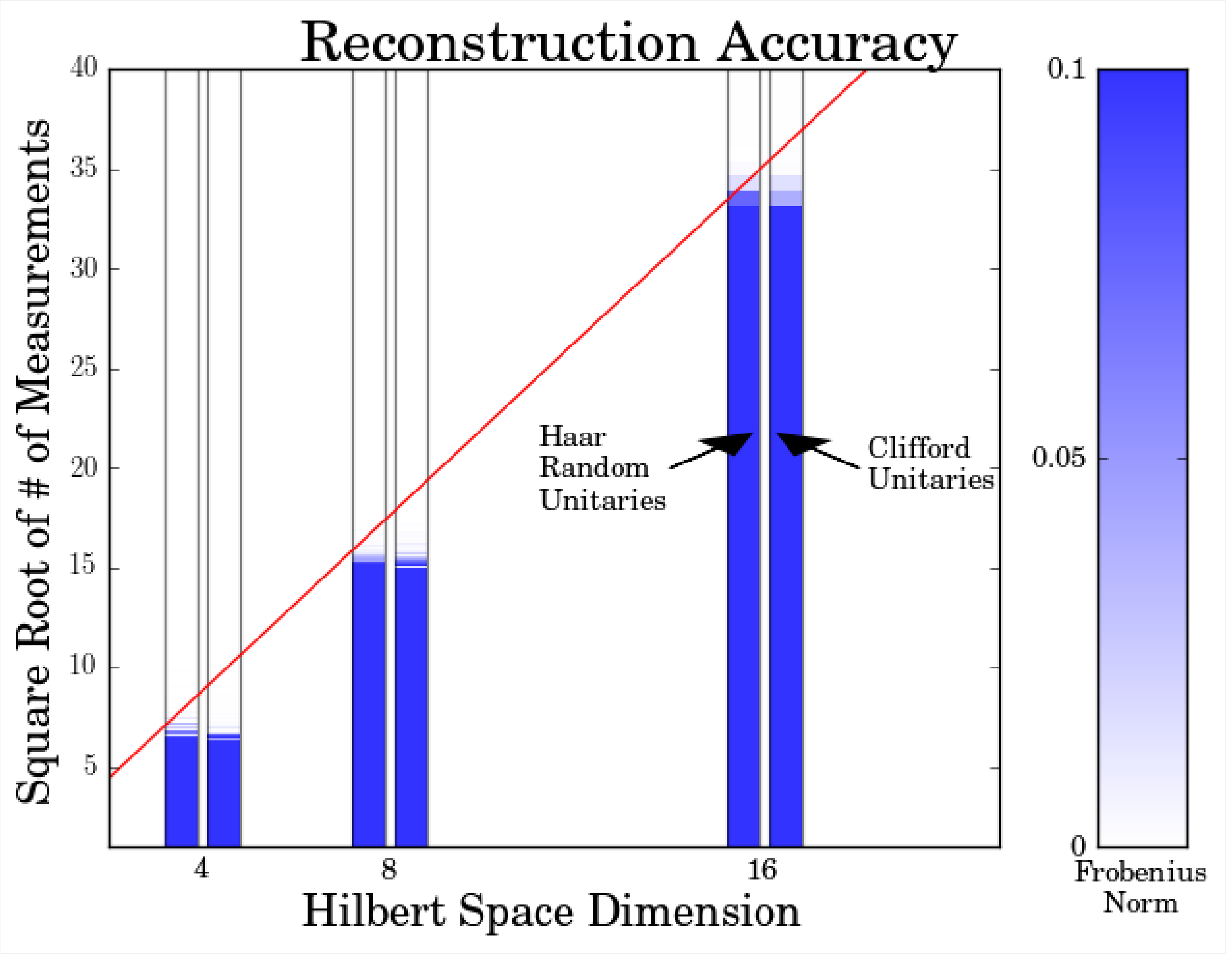

Summary and Conclusions

- Phase retrieval of unitary matrices
  - Motivation: quantum gate tomography
  - Used variant of PhaseLift algorithm
- Exact recovery using unitary 4-designs, approximate recovery using unitary 2-designs. See **arXiv/1510.08887**
- Outlook:
  - What about 3-designs? What about Cliffords in particular?
  - Different algorithm, e.g. Wirtinger Flow?

PhaseLift for Unitary Matrices

- Measurements $y = \mathcal{A}\big(\mathrm{vec}(U)\mathrm{vec}(U)^\dagger\big) + \varepsilon$
  - Where

$$\mathcal{A} : \mathbb{C}^{d^2\times d^2}_{\mathrm{Herm}} \to \mathbb{R}^m$$

$$\mathcal{A}(\Gamma) = \left[\mathrm{vec}(\sqrt{d}C_i)^\dagger \Gamma \mathrm{vec}(\sqrt{d}C_i)\right]_{i=1}^m$$

- Convex program

$$\begin{aligned} \arg \min_{\Gamma \in \mathbb{C}^{d^2\times d^2}_{\mathrm{Herm}}} \ & \mathrm{tr}(\Gamma) \text{ such that} \\ & \|\mathcal{A}(\Gamma) - y\|_2 \le \eta, \\ & \Gamma \succeq 0, \\ & \mathrm{tr}_1(\Gamma) = (I/d)\,\mathrm{tr}(\Gamma), \\ & \mathrm{tr}_2(\Gamma) = (I/d)\,\mathrm{tr}(\Gamma). \end{aligned}$$

PhaseLift Using Unitary 4-Designs

- “Exact” recovery guarantee:

Suppose that the number of measurements satisfies

$$m \geq \left(64(4!)^2 c_0\right)^2 \cdot d^2 \ln d. \qquad \text{(I.11)}$$

Then with probability at least $1 - \exp\left(-2m\,(4(4!))^{-4}\right)$ *(over the choice of the* C_i*), we have the following uniform recovery guarantee:*

For any unitary matrix $U \in \mathbb{C}^{d\times d}$*, it is the case that any solution* Γ_{opt} *to the convex program (I.7) with noisy measurements (I.6) must satisfy:*

$$\|\Gamma_{opt} - vec(U)vec(U)^\dagger\|_F \leq \frac{128(4!)^2\eta}{\sqrt{m}}\left(1 + \frac{2c_5}{c_0 - 2c_5}\right). \qquad \text{(I.12)}$$

PhaseLift Using Unitary 2-Designs

- We will seek to recover all unitary matrices U that are "non-spiky" with respect to the measurement matrices C_i

Let $\tilde{G}$ be a finite set of unitary matrices in $\mathbb{C}^{d\times d}$. We say that a unitary matrix $U \in \mathbb{C}^{d\times d}$ is *non-spiky* with respect to $\tilde{G}$ (with parameter $\beta \geq 0$) if the following holds:

$$|\mathrm{tr}(C^{\dagger}U)|^2 \leq \beta, \quad \forall C \in \tilde{G}. \tag{I.13}$$

- Fact: Almost all unitary matrices are non-spiky, with parameter $\beta \sim \log|\tilde{G}|$

PhaseLift Using Unitary 2-Designs

- For all β-non-spiky unitary matrices U, we achieve "approximate" recovery, up to error δ
  - Let v = β/δ

Suppose that the number of measurements satisfies

$$m \geq \left(8c_0\nu^2\right)^2 \cdot d^2 \ln d. \qquad \text{(I.16)}$$

Then with probability at least $1 - \exp(-\frac{1}{128}m\nu^{-4})$ *(over the choice of the* C_i*), we have the following uniform recovery guarantee:*

For any unitary matrix $U \in \mathbb{C}^{d\times d}$ *that is non-spiky with respect to* $\tilde{G}$ *(with parameter* β*, in the sense of (I.13)),* [redacted]

$$\|\Gamma_{opt} - vec(U)vec(U)^\dagger\|_F \leq \max\left\{\delta\|vec(U)vec(U)^\dagger\|_F,\ \frac{16\eta\nu^2}{\sqrt{m}}\left(1 + \frac{2c_5}{c_0 - 2c_5}\right)\right\}. \qquad \text{(I.17)}$$