Path Detection:
A Quantum Computing Primitive

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Based on work with
Stacey Jeffery: arXiv: 1704.00765
Michael Jarret, Stacey Jeffery, Alvaro Piedrafita, *in progress*

Middlebury

How to make quantum algorithms accessible?

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- New primitive: ***st*-connectivity**

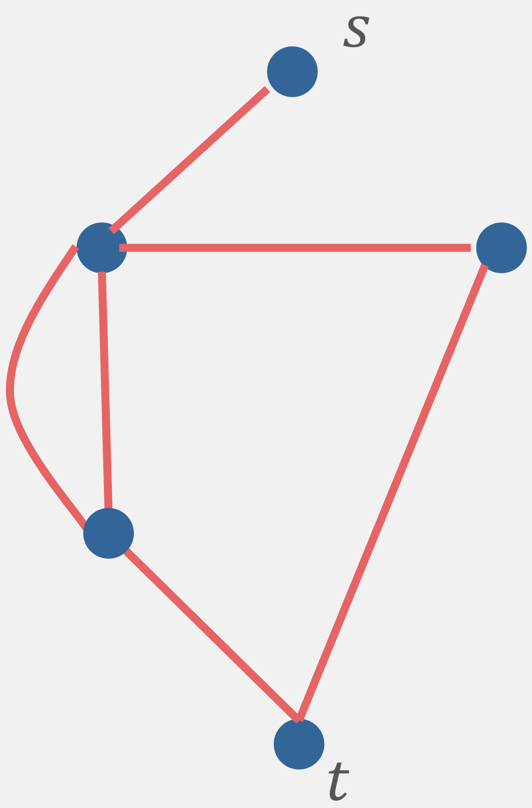
Outline:

A. Introduction to st-connectivity
B. st-connectivity makes a good algorithmic primitive
   1. Applies to a wide range of problems

   2. Easy to understand (without knowing quantum mechanics)
C. Applications and performance of algorithm

st-connectivity

$\boldsymbol{st - connectivity}$:
is there a path from s to t?



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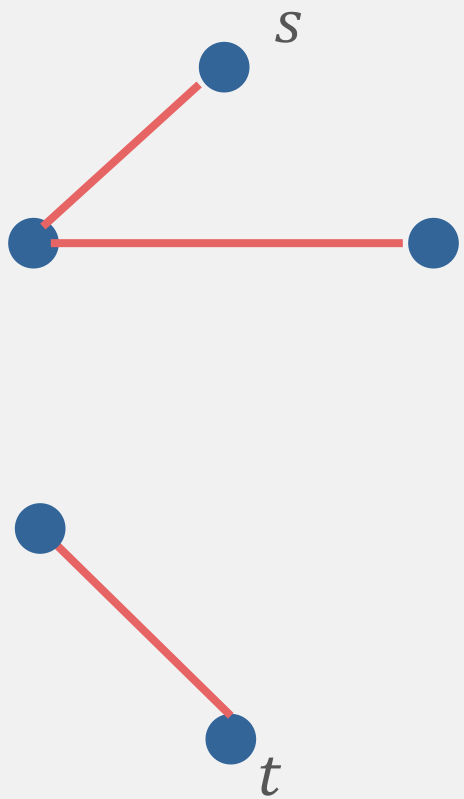
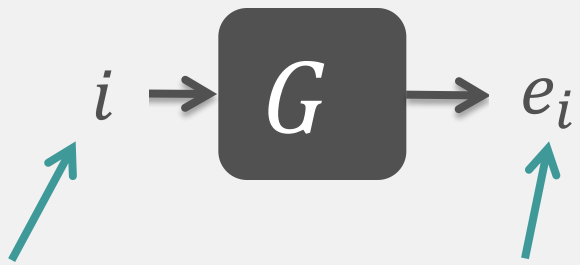
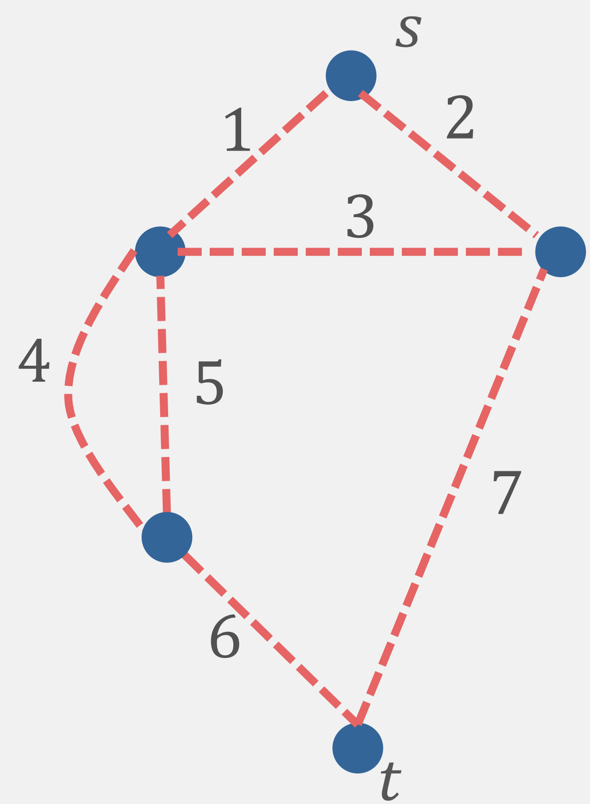


Black Box Model



Edge label

- $e_i = 1$ if i^{th} edge is there
- $e_i = 0$ if edge is not there



Let $\mathcal{H}$ be the set of graphs G that the black box might contain.

Figure of Merit

- Query Complexity
  - Number of uses (queries) of the black box
  - All other operations are free

- Under mild assumption, for our algorithm,

  quantum query complexity $\cong$ quantum time complexity

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Outline:

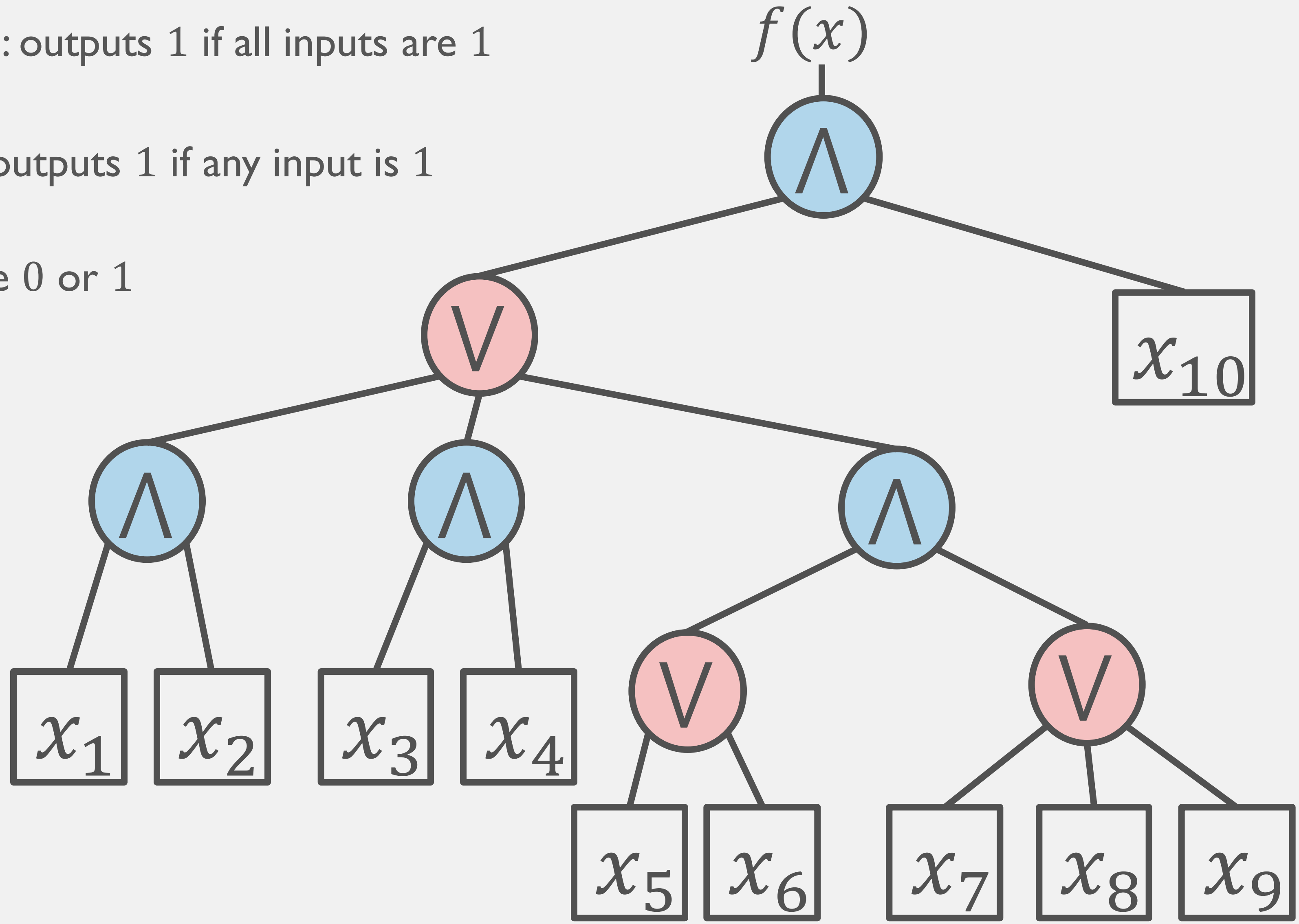
A. Introduction to st-connectivity
B. st-connectivity makes a good algorithmic primitive
    1. Applies to a wide range of problems
        - Evaluating Boolean formulas reduces to st-connectivity
    2. Easy to understand (without knowing quantum mechanics)
C. Applications and performance of algorithm

Boolean Formulas

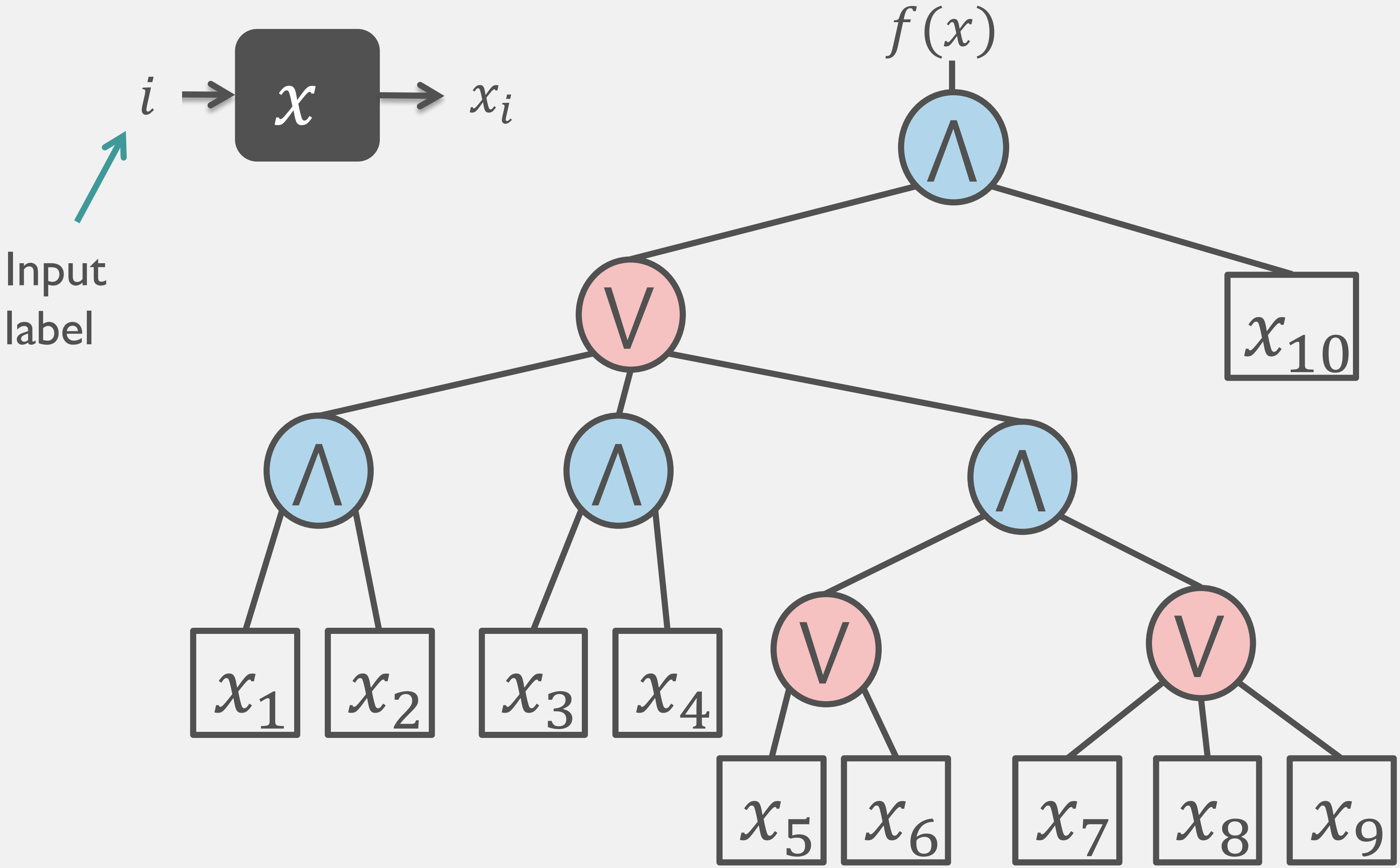
$\wedge$ *AND*: outputs 1 if all inputs are 1

$\vee$ *OR*: outputs 1 if any input is 1

x_i Value 0 or 1

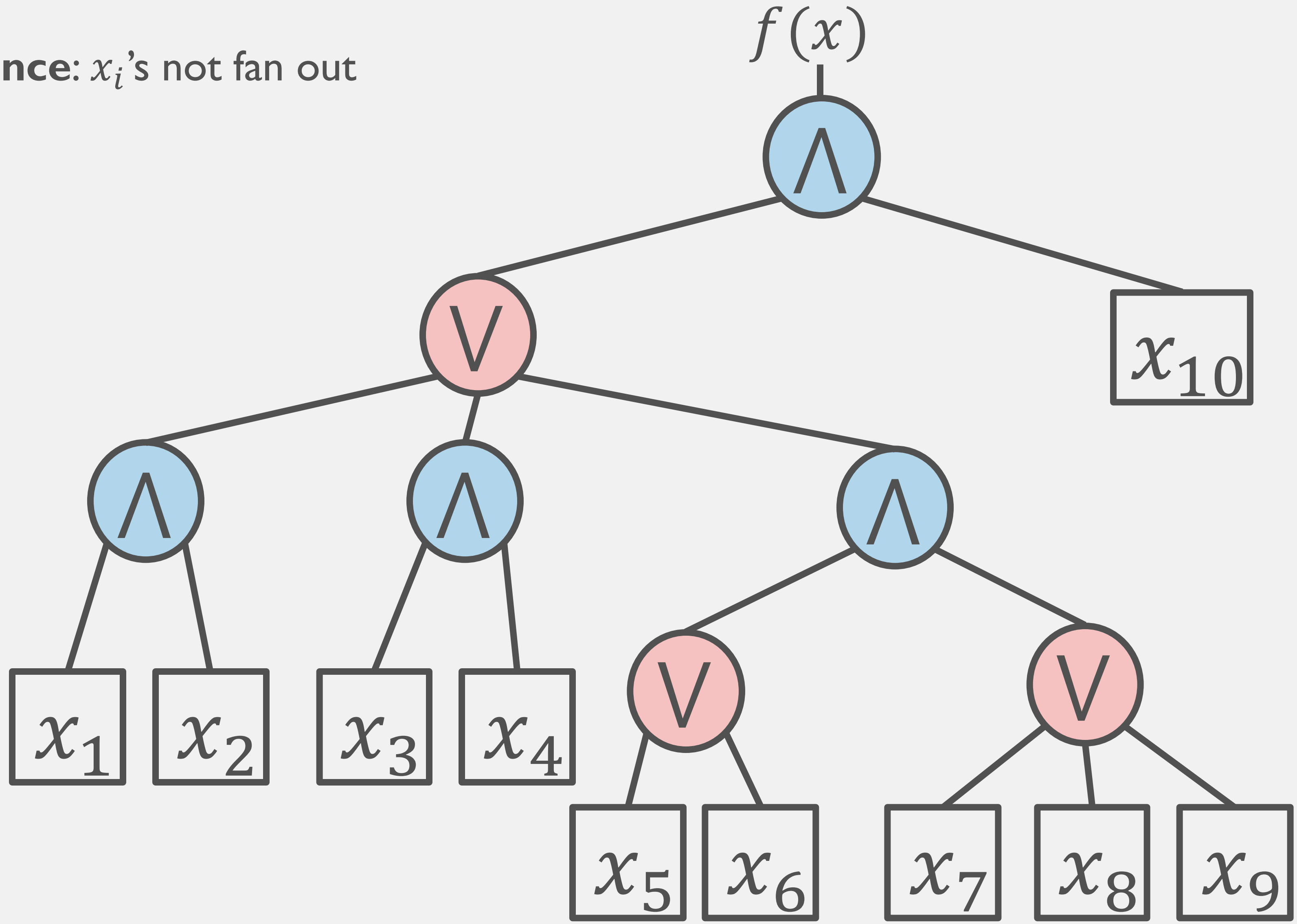


Boolean Formulas



Boolean Formulas

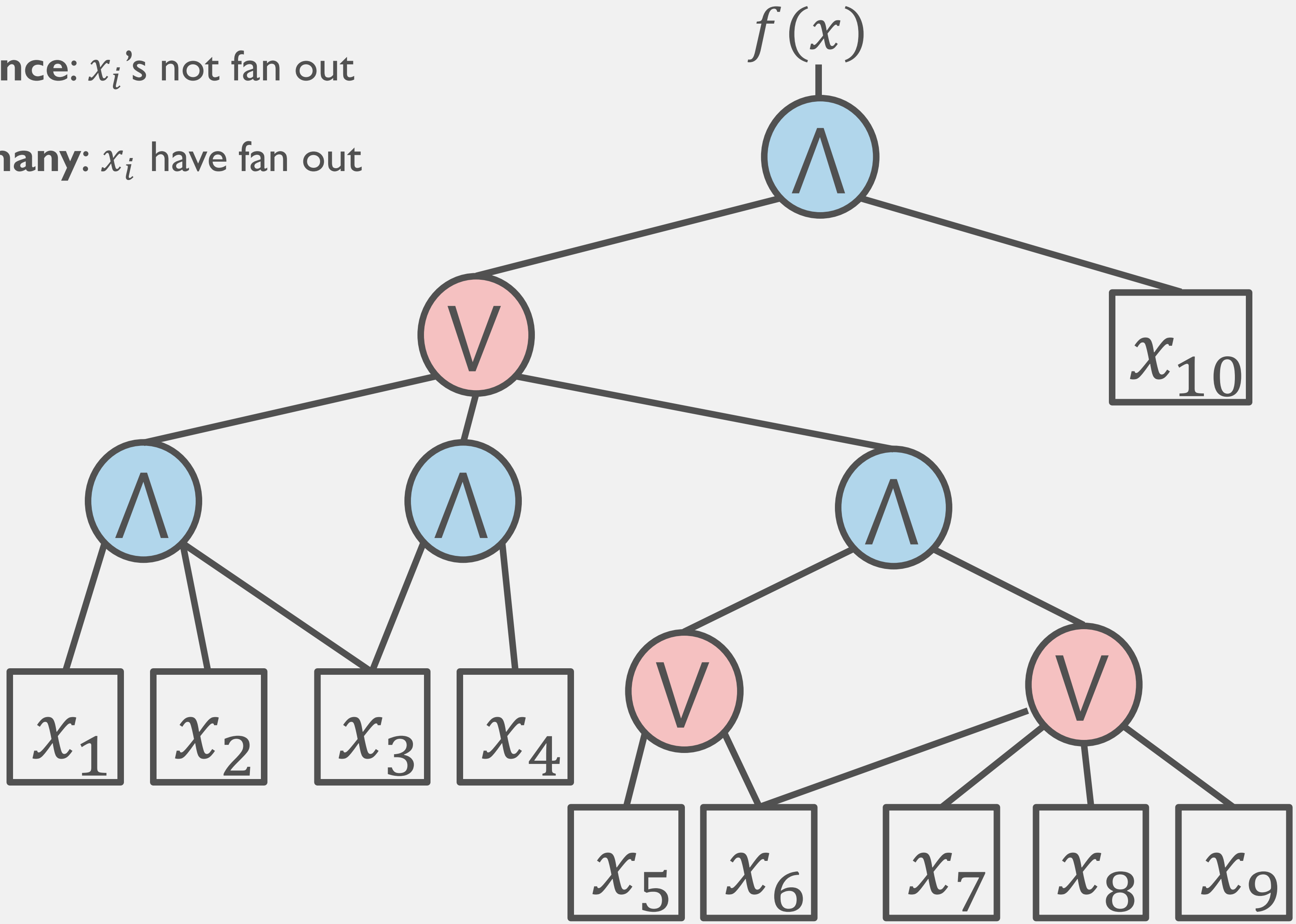
Read-once: x_i's not fan out



Boolean Formulas

Read-once: x_i's not fan out

Read-many: x_i have fan out

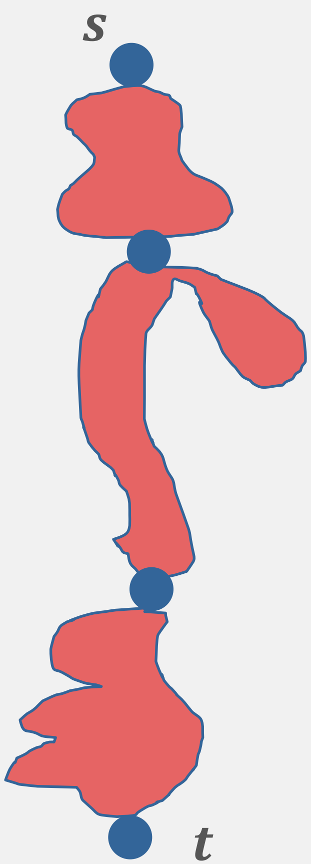


Boolean Formula Applications

- Logic
- Designing electrical circuits
- Game theory (deciding who will win a game)
- Combinatorics and graph problems
- Linear programming
- Testing potential solution to an NP-complete problem

Application to Boolean Formulas

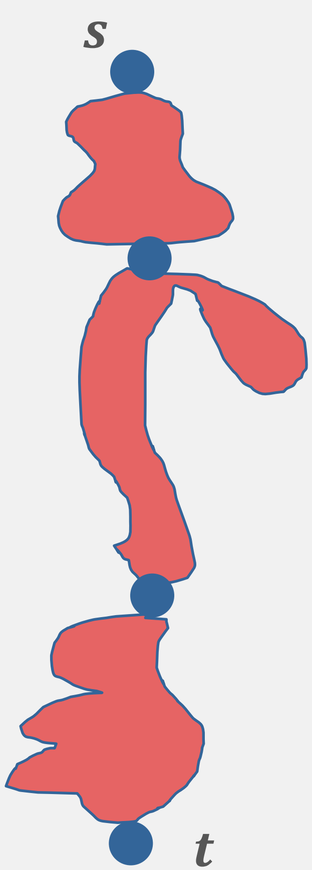
AND: outputs 1 if all input subformulas have value 1



s and t are connected if all subgraphs are connected

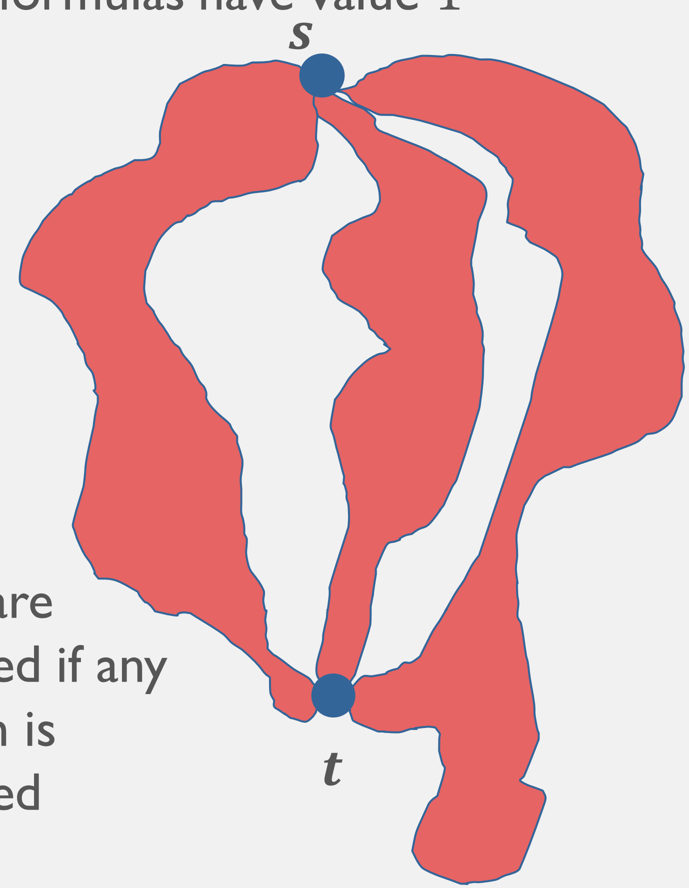
Application to Boolean Formulas

$\wedge$ AND: outputs 1 if all input subformulas have value 1



s and t are connected if all subgraphs are connected

$\vee$ OR: outputs 1 if any input subformulas have value 1



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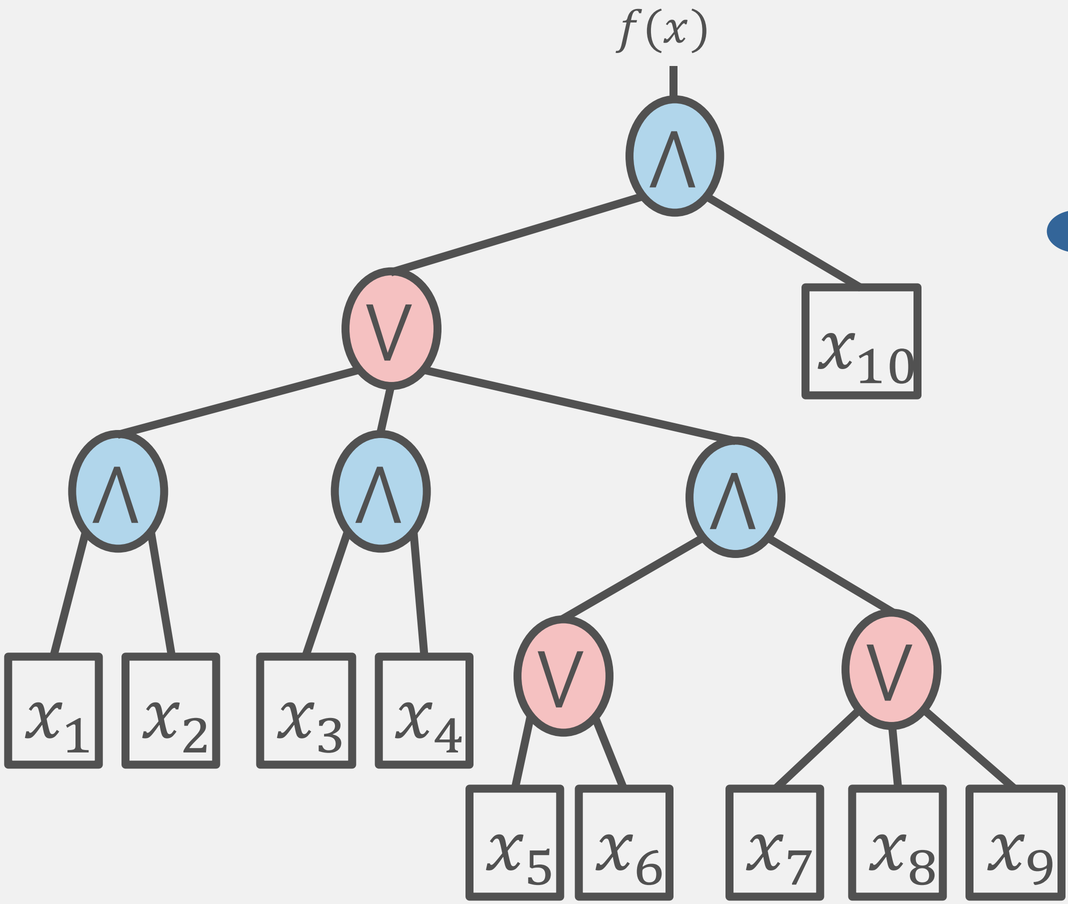
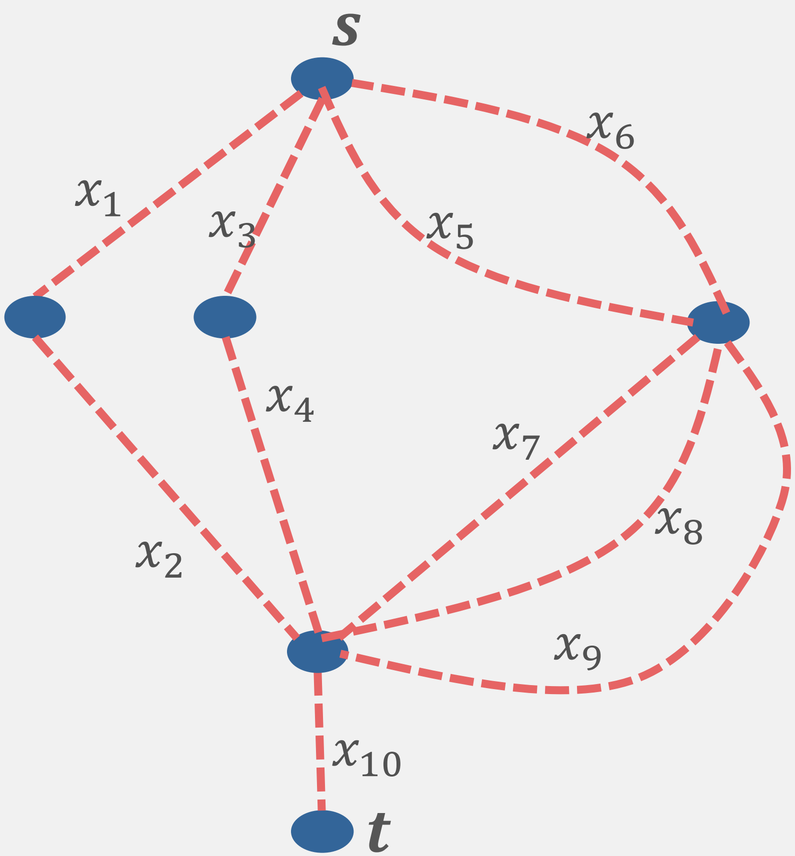
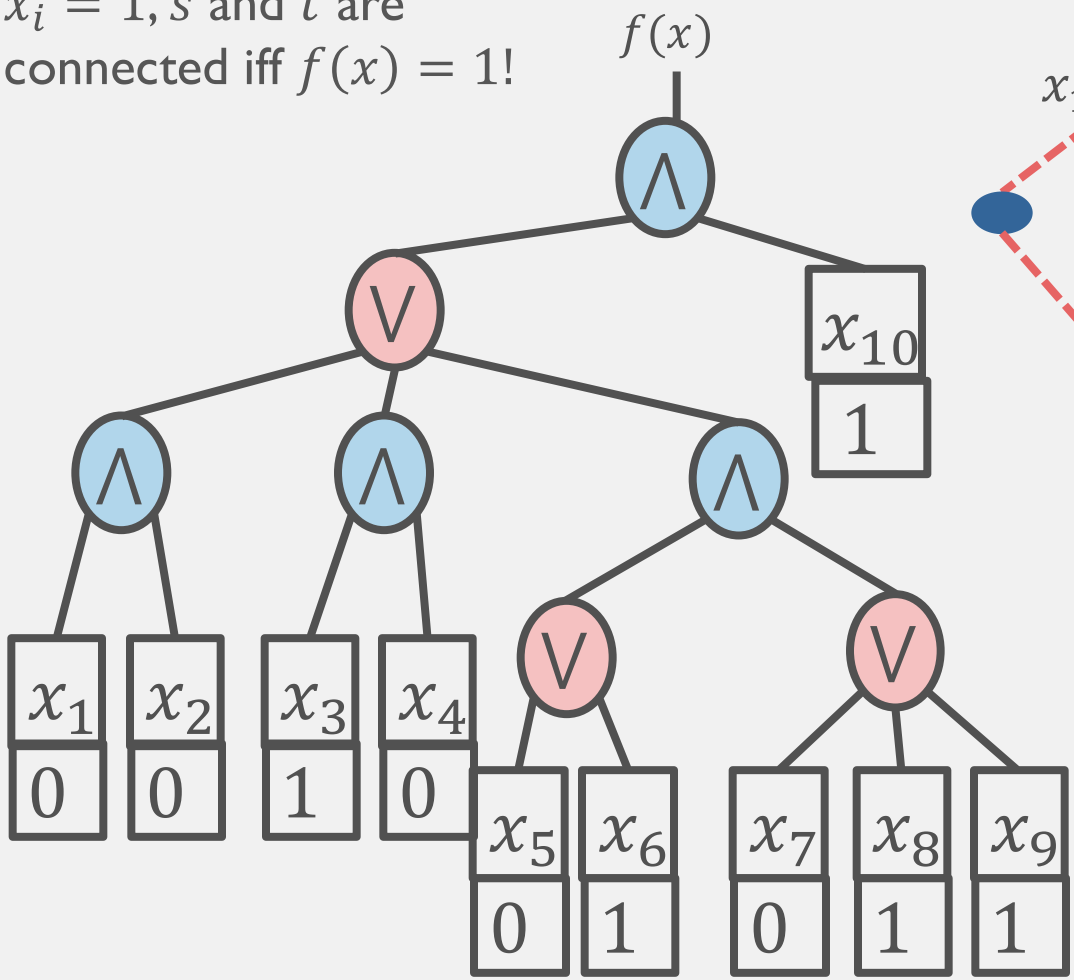
Application to Boolean Formulas

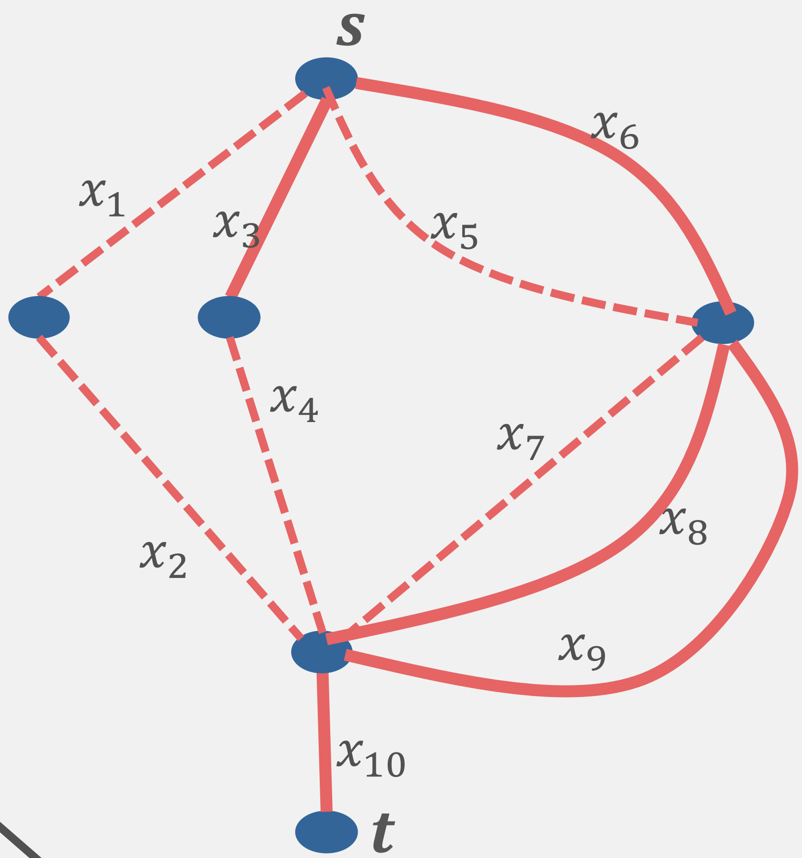




Application to Boolean Formulas

- If we put edges where $x_i = 1$, s and t are connected iff $f(x) = 1$!





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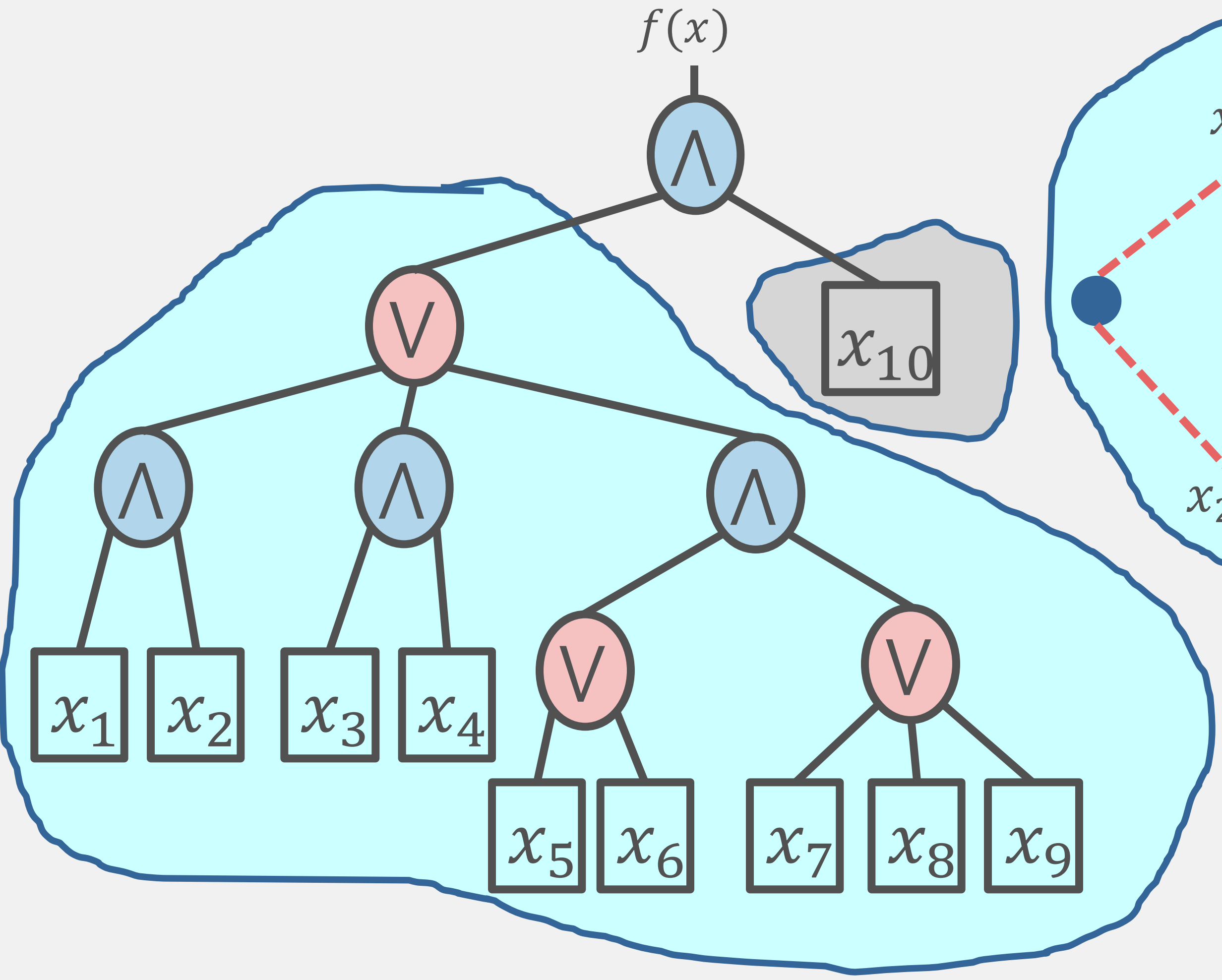

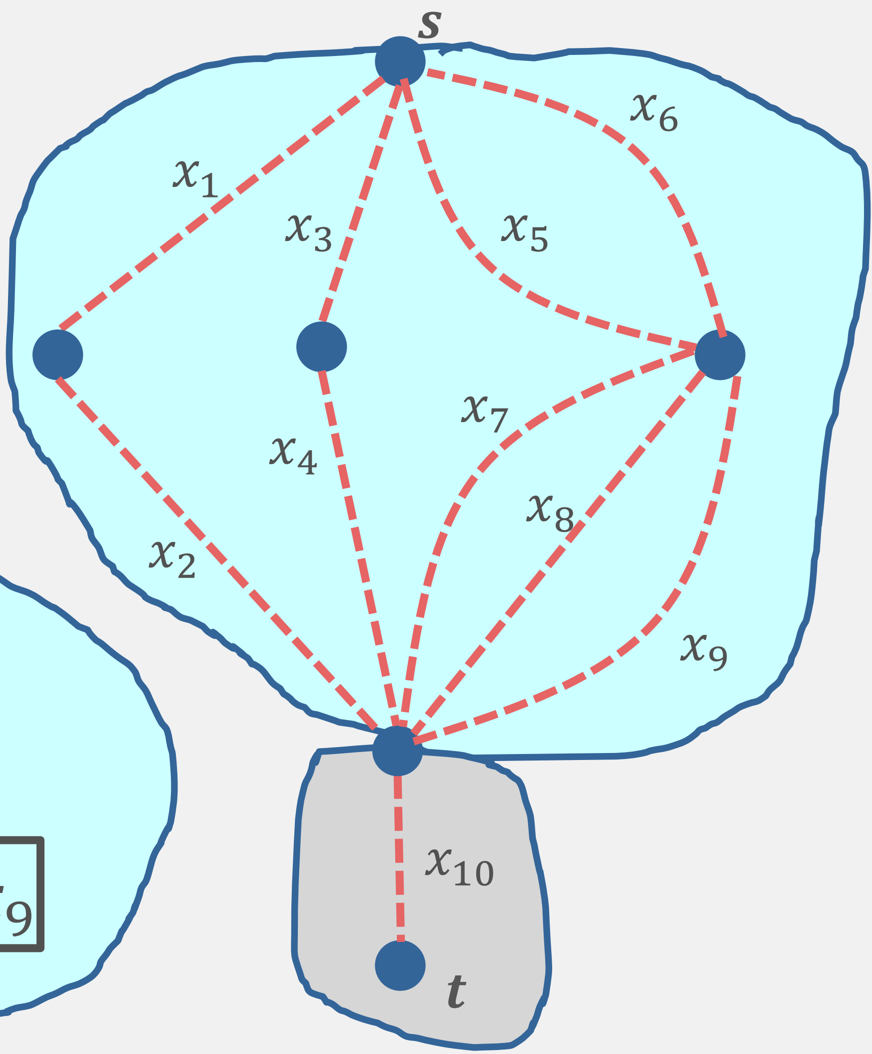

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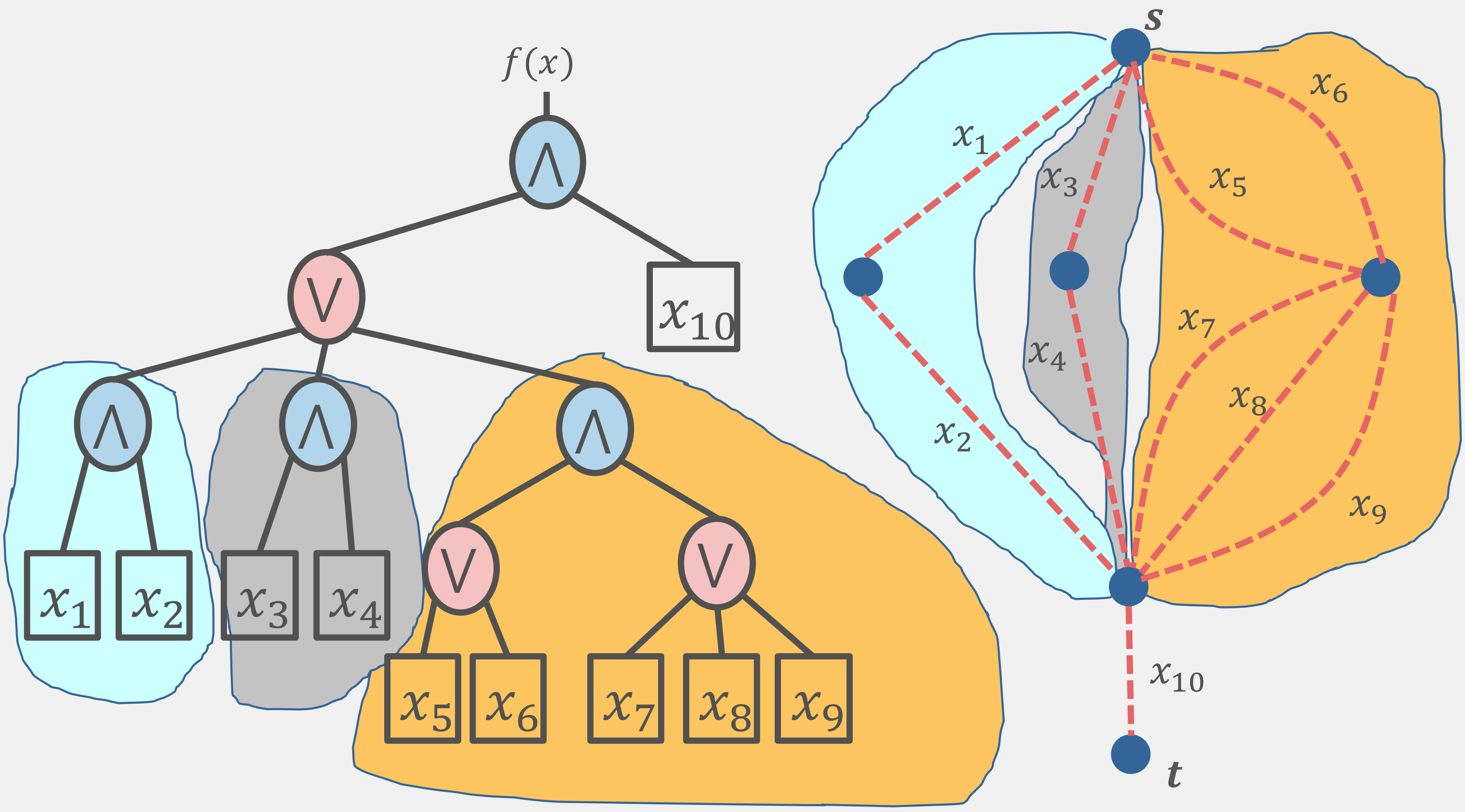

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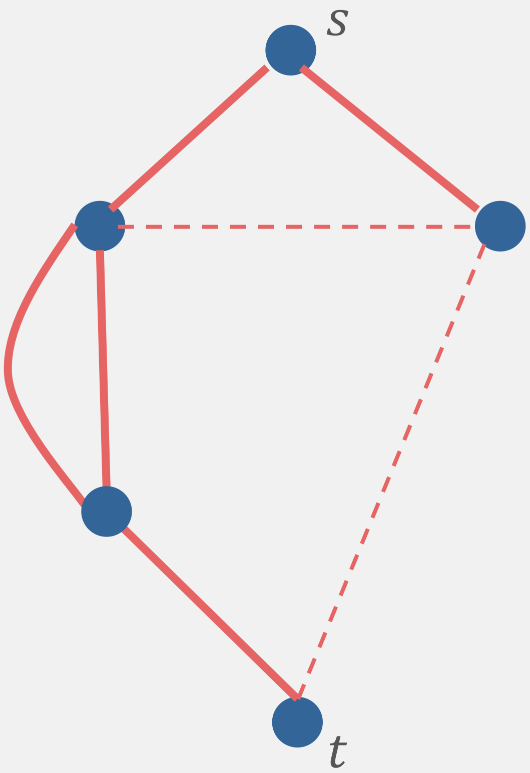
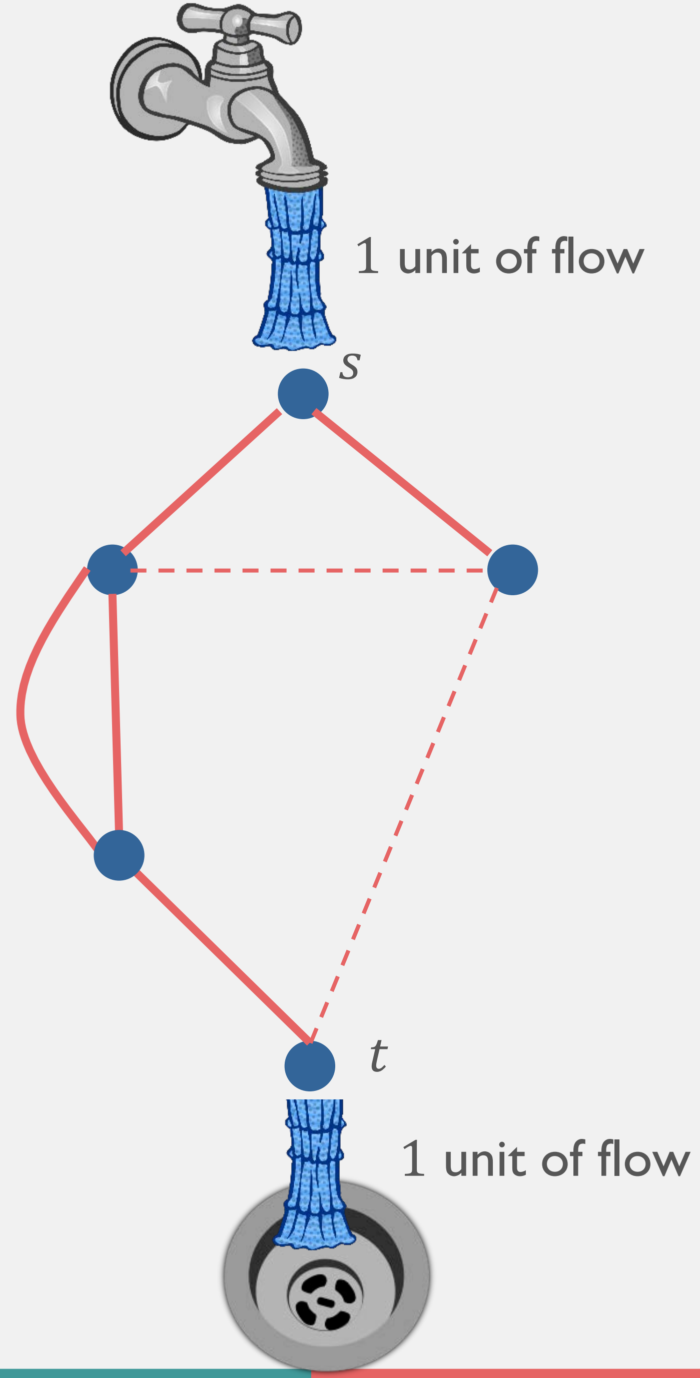
Effective Resistance

Graph G:



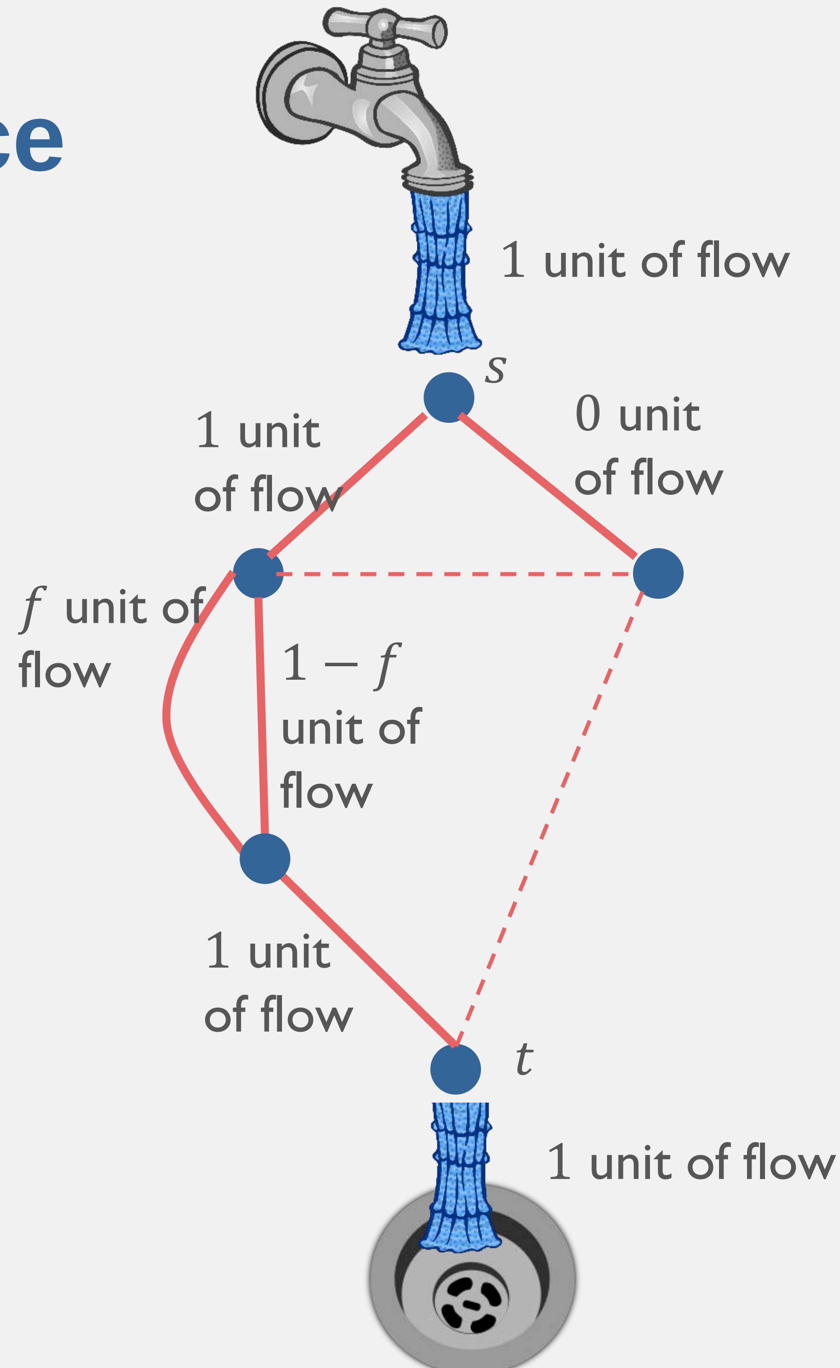
Effective Resistance



Effective Resistance

Valid flow:

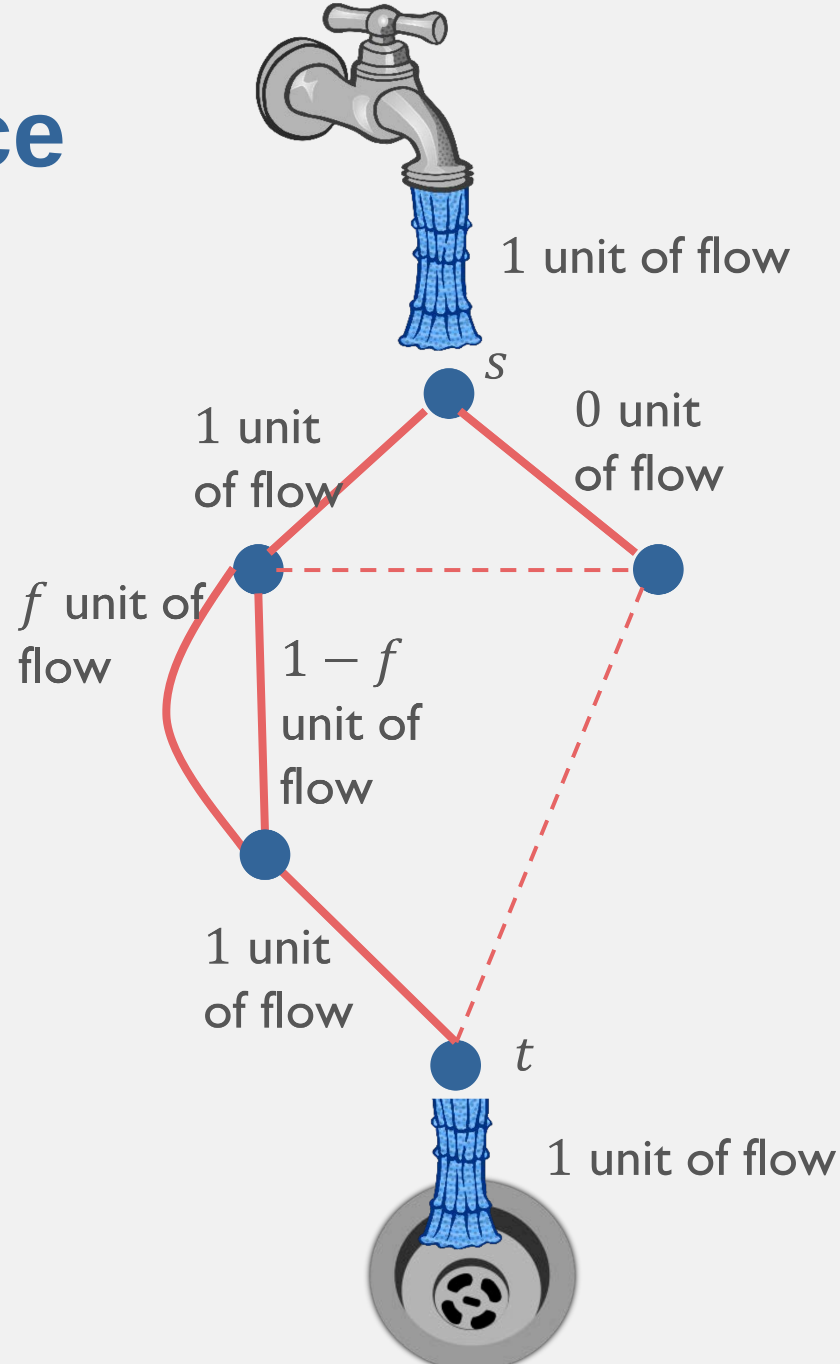
- 1 unit in at s
- 1 unit out at t
- At all other nodes, zero net flow



Effective Resistance

Flow energy:

$$\sum_{edges} (flow\ on\ edge)^2$$



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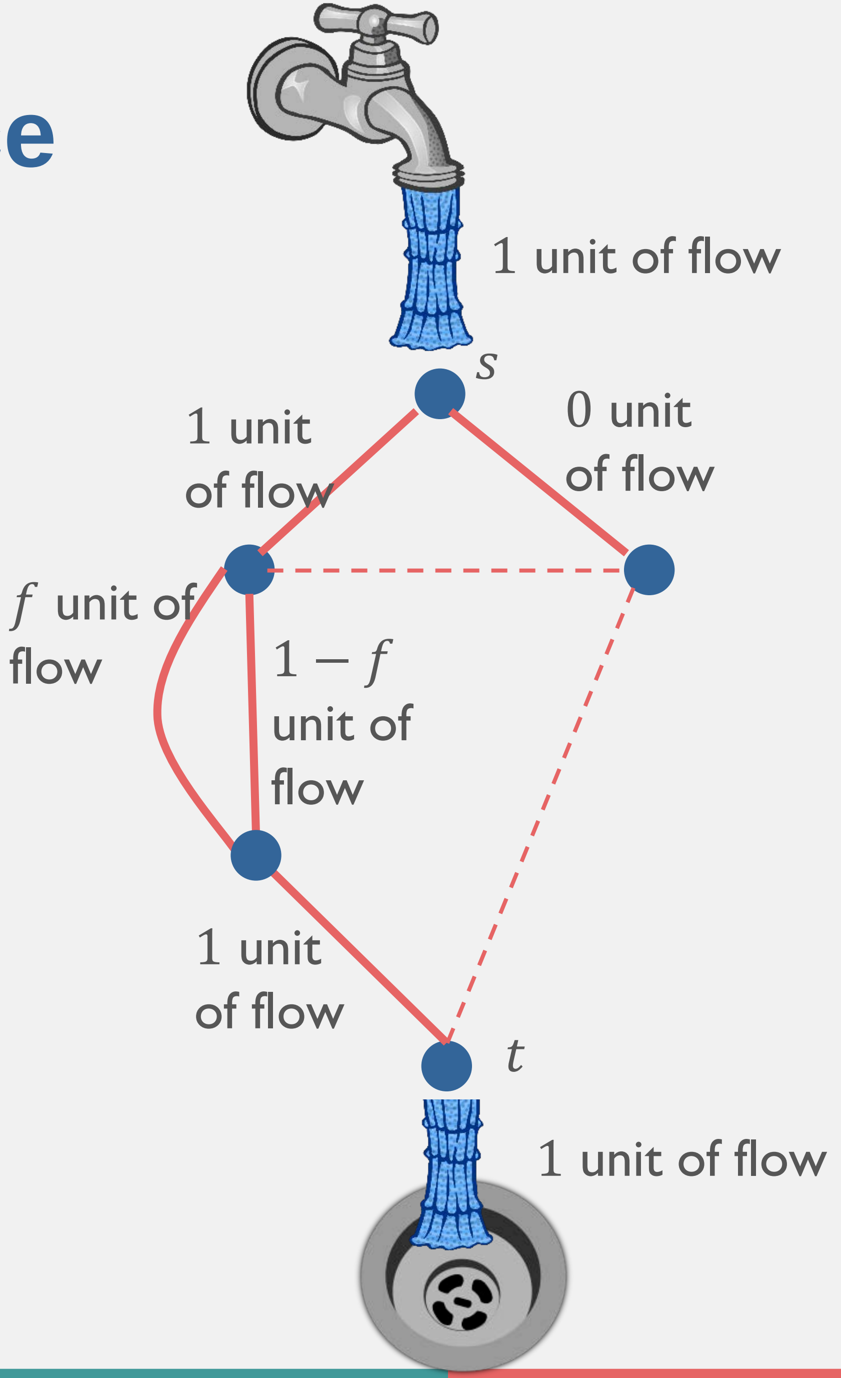
Effective Resistance: $R_{s,t}(G)$

Smallest energy of any valid flow from s to t on G.



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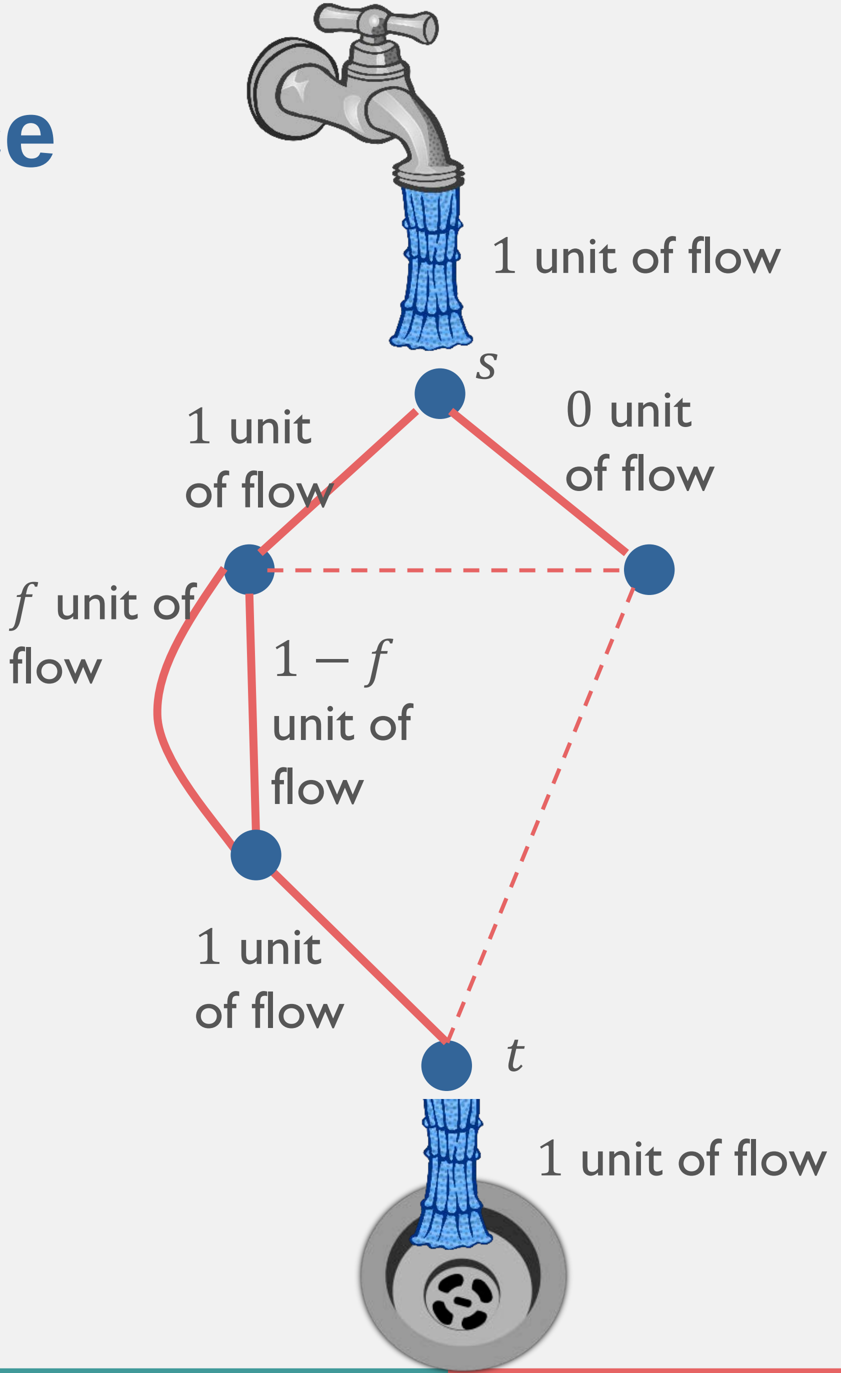
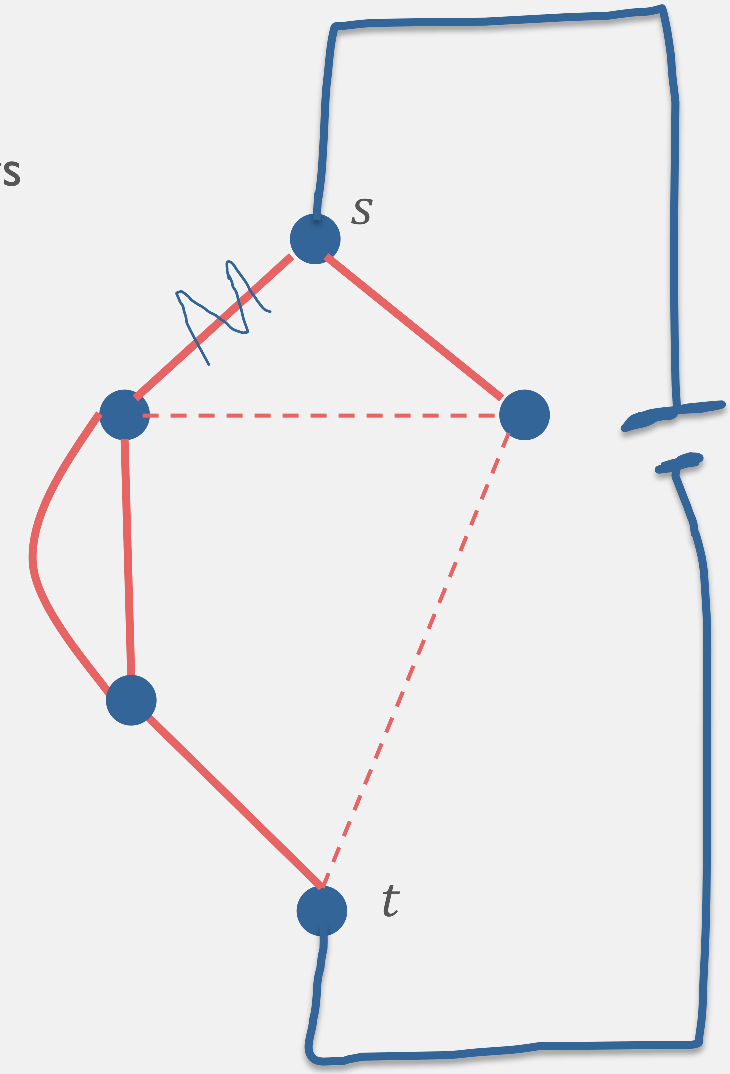
Properties of $R_{s,t}(G)$

- Small if many short paths from s to t
- Large if few long paths from s to t
- Infinite if s and t not connected



Effective Resistance

1 unit resistors



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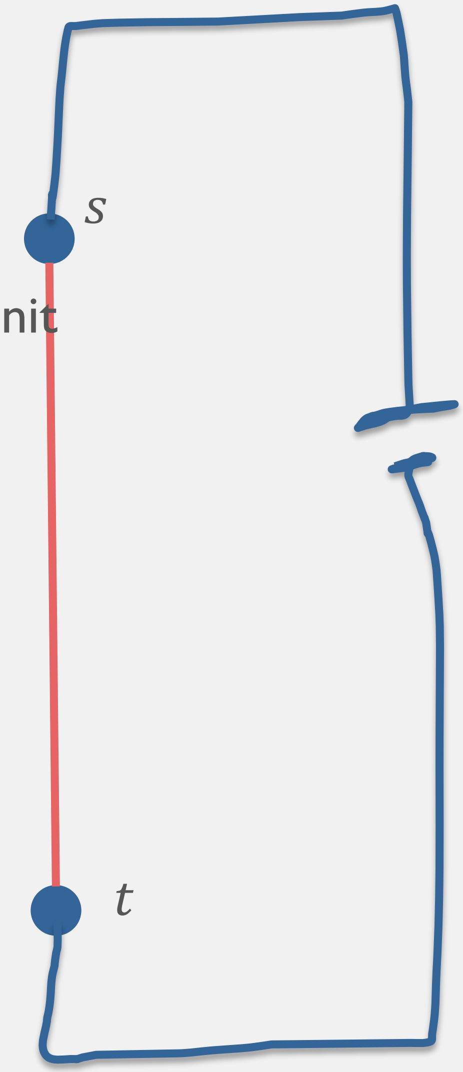
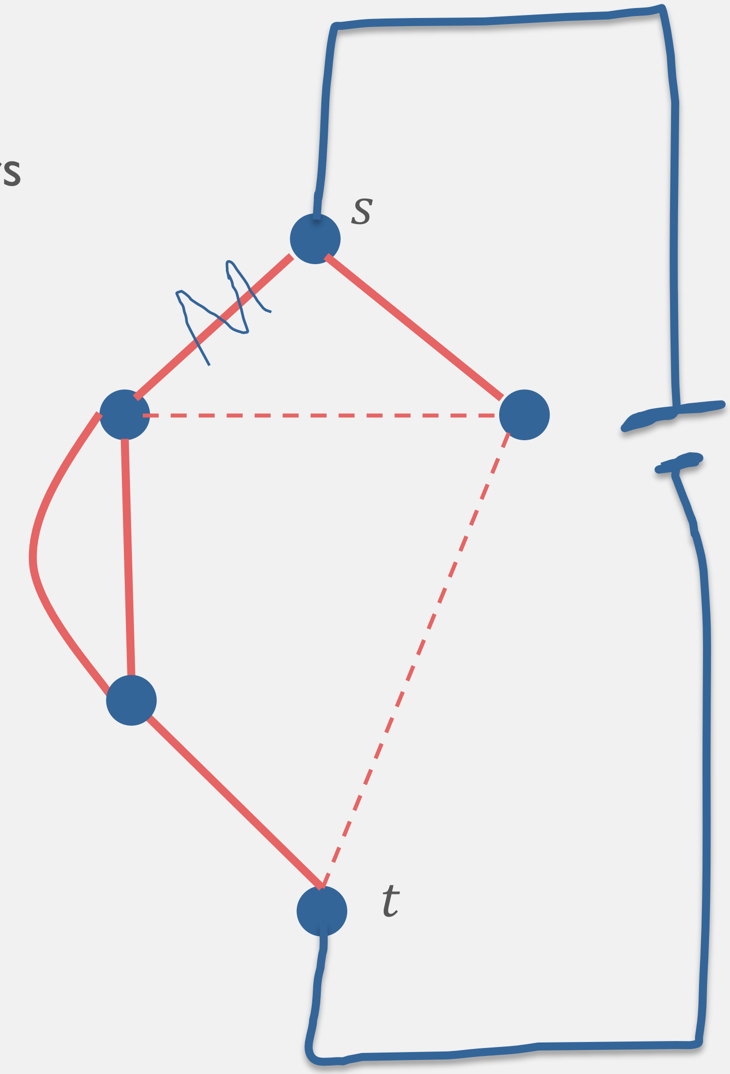


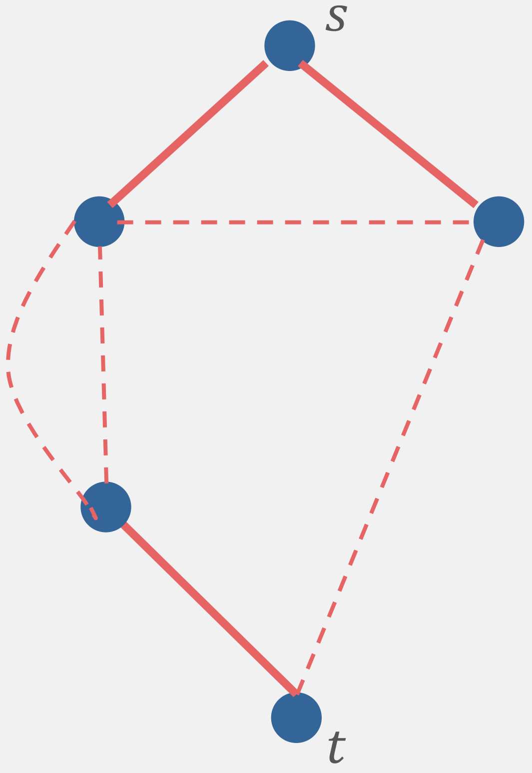
$R_{s,t}(G)$ unit resistor

1 unit resistors



Effective Capacitance

Graph G':

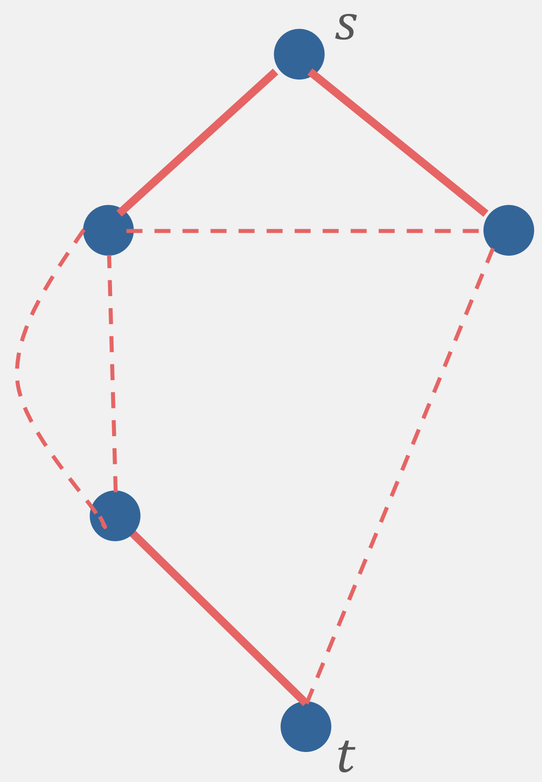


Effective Capacitance

Valid potential energy:

- 1 at s
- 0 at t
- Potential energy difference is 0 across edge

Graph G':



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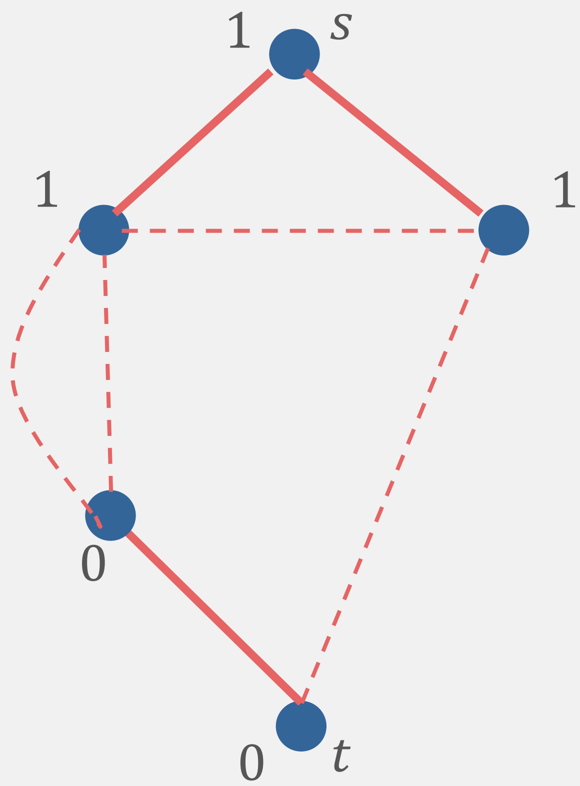


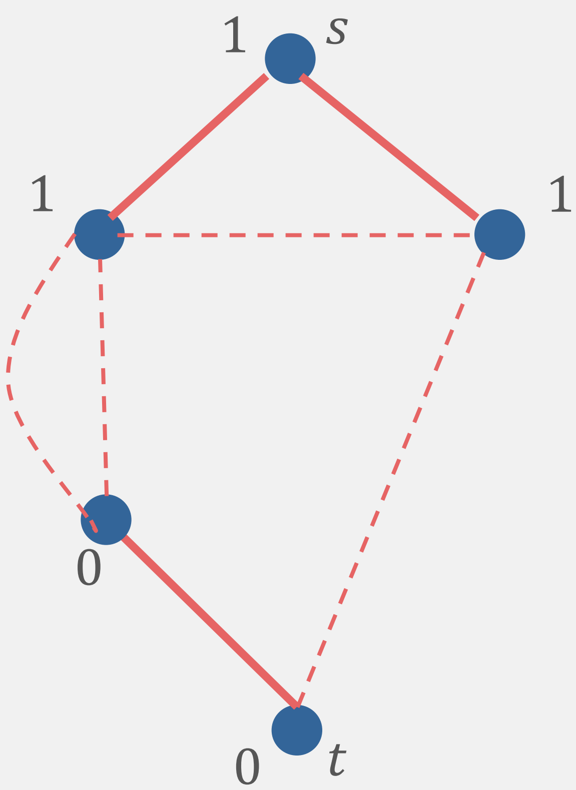
Effective Capacitance

Cut energy:

$$\sum_{edges} (Potential\ Energy\ Difference)^2$$

Effective Capacitance: $C_{s,t}(G')$
Smallest cut energy of any valid potential energy between s to t on G'.

Graph G':



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Properties of $C_{s,t}(G')$
- Small if many small cuts
- Large if one large cuts
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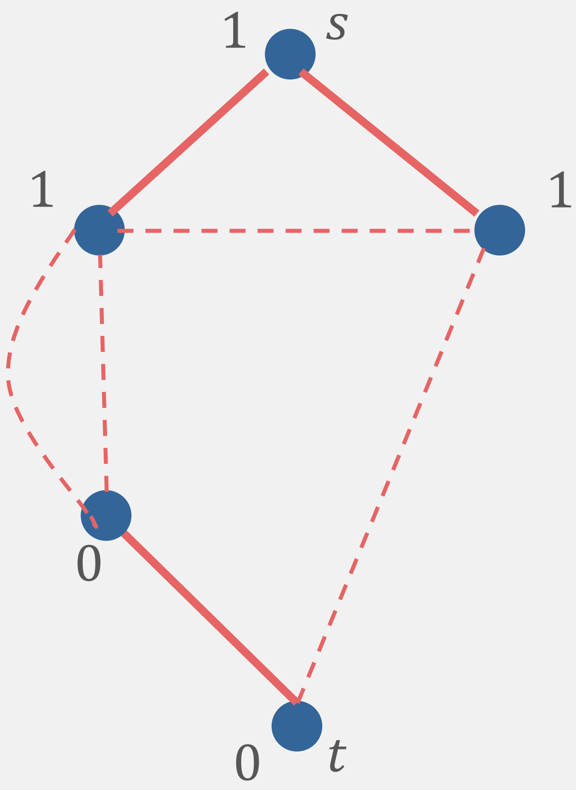
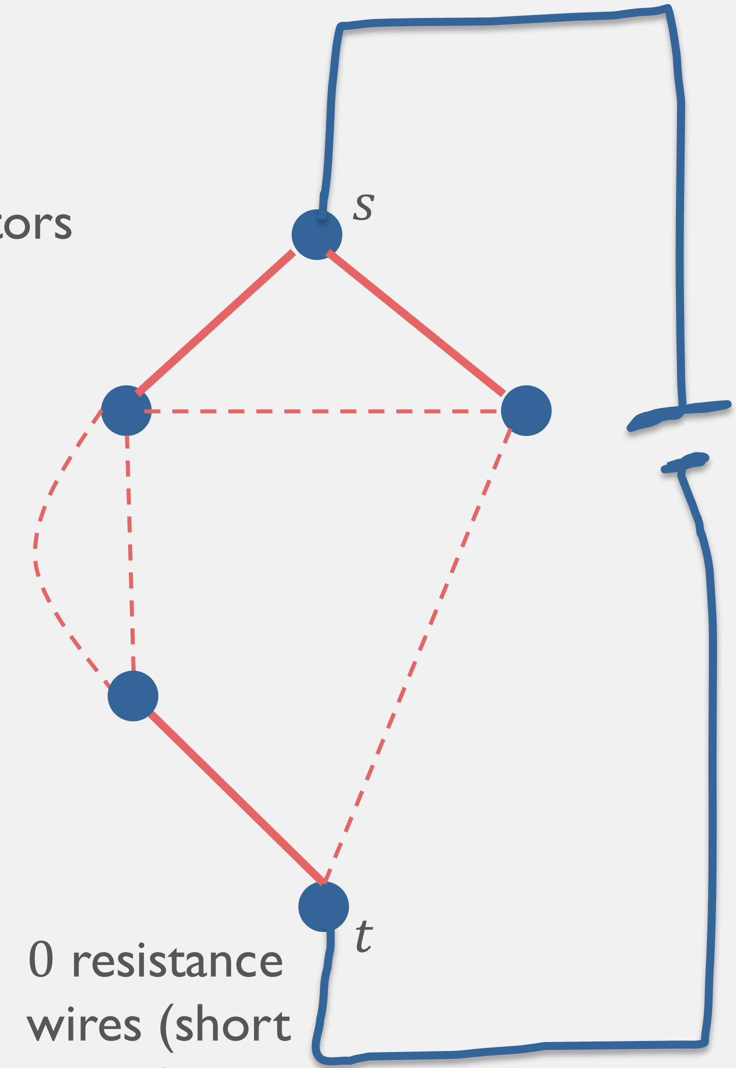
Graph G':



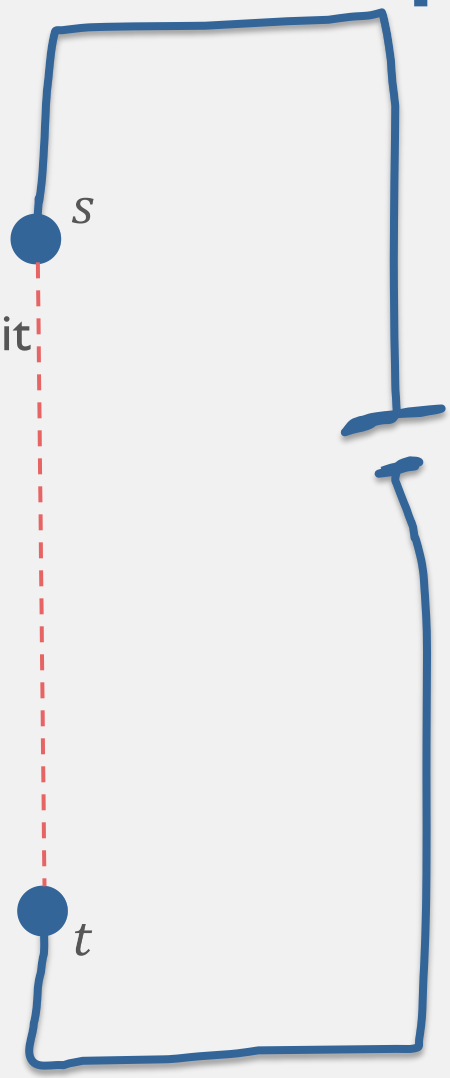
Effective Capacitance

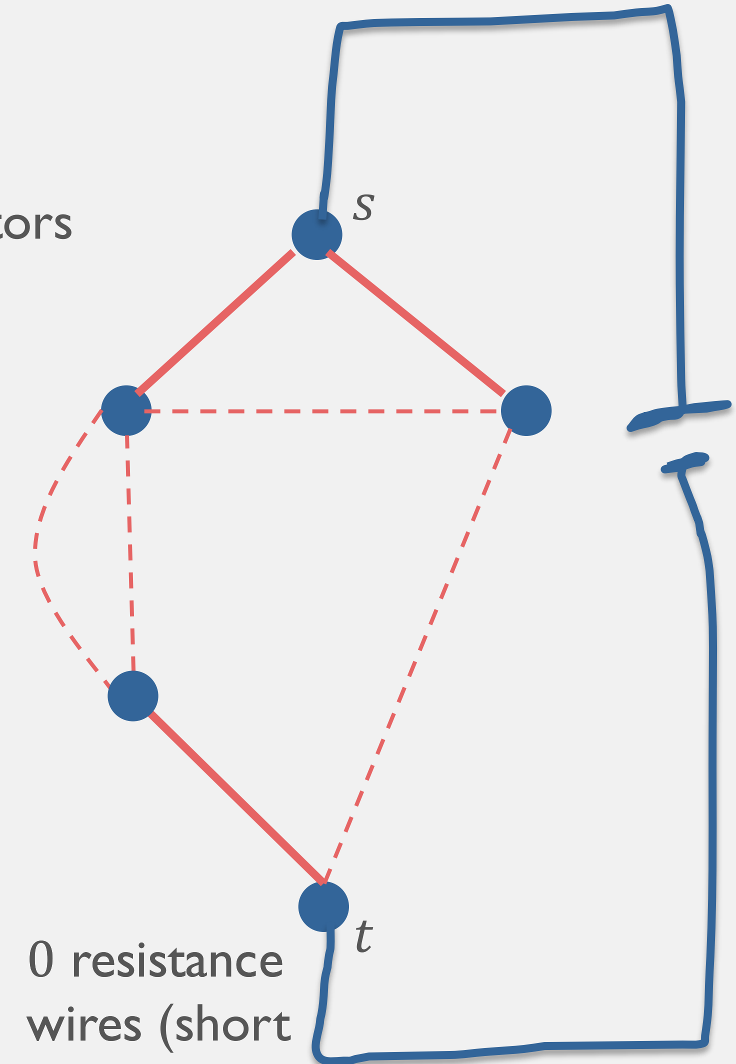
1 unit capacitors



0 resistance wires (short circuit)

Effective Capacitance





Algorithm Performance:

st-connectivity algorithm complexity =

$$O\left(\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)}\ \sqrt{\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G')}\right)$$

† with (s,t) added also planar

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[Belovs, Reichard, '12] [JJ**K**P, in progress]

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Example

What is quantum complexity of deciding $AND(x_1, x_2, \ldots, x_N)$, promised

- All $x_i = 1$, or
- At least $\sqrt{N}$ input variables are 0.

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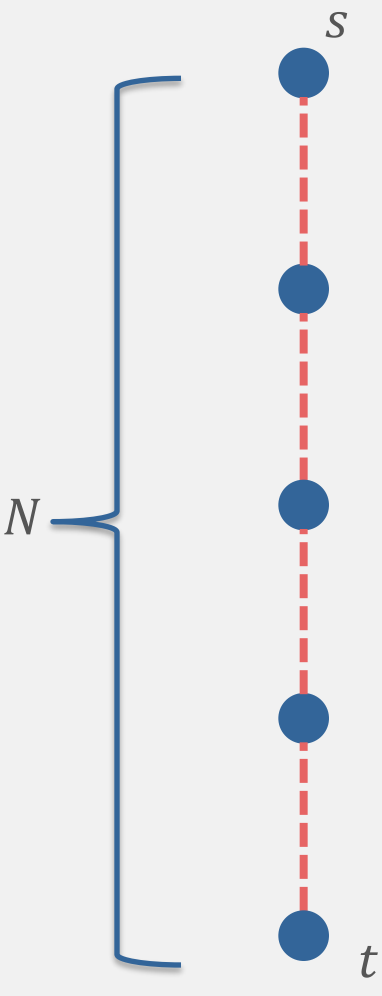

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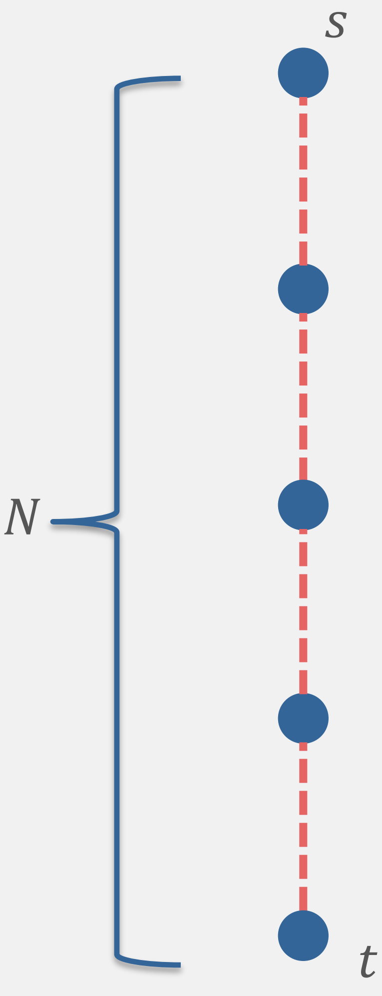

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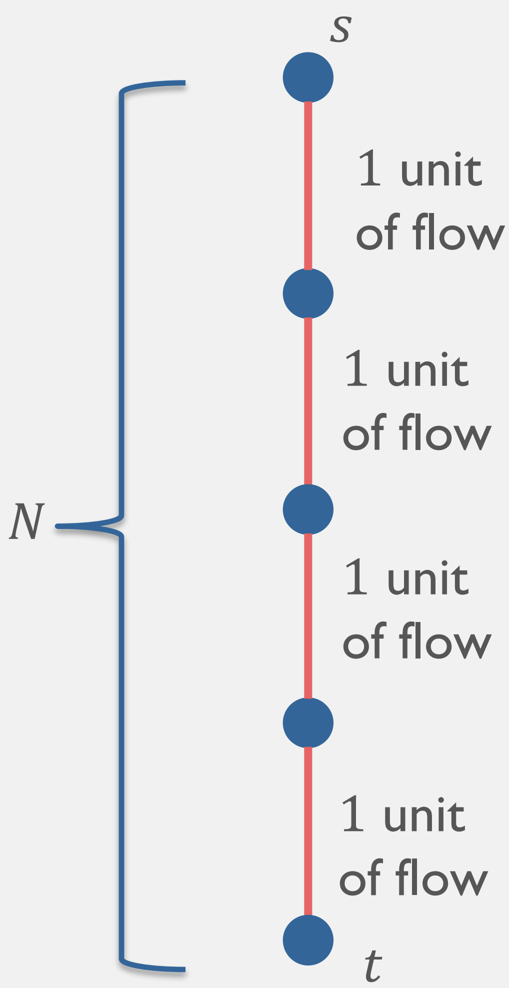

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$$\max_{G\in\mathcal{H}:connected} R_{s,t}(G) = N$$

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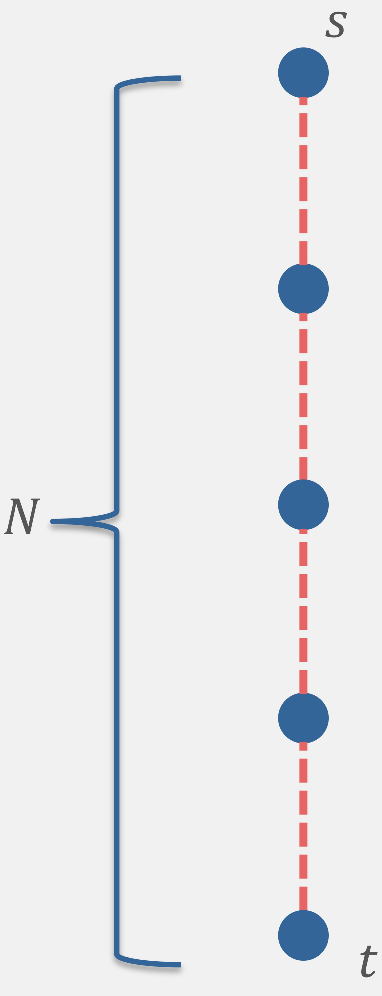

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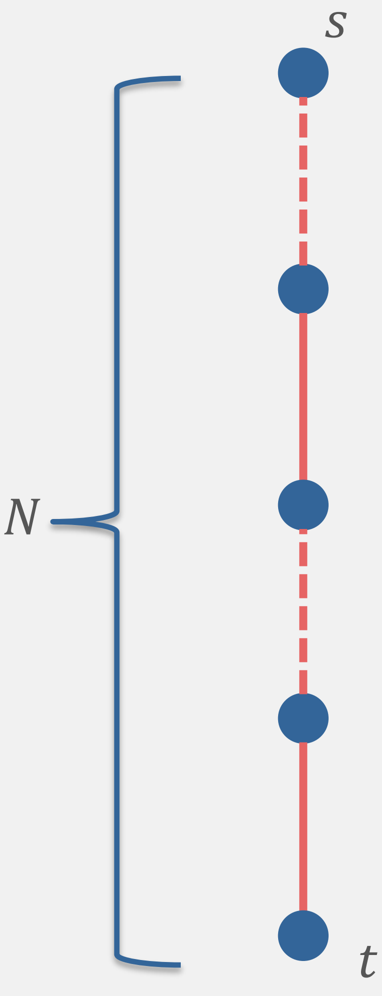

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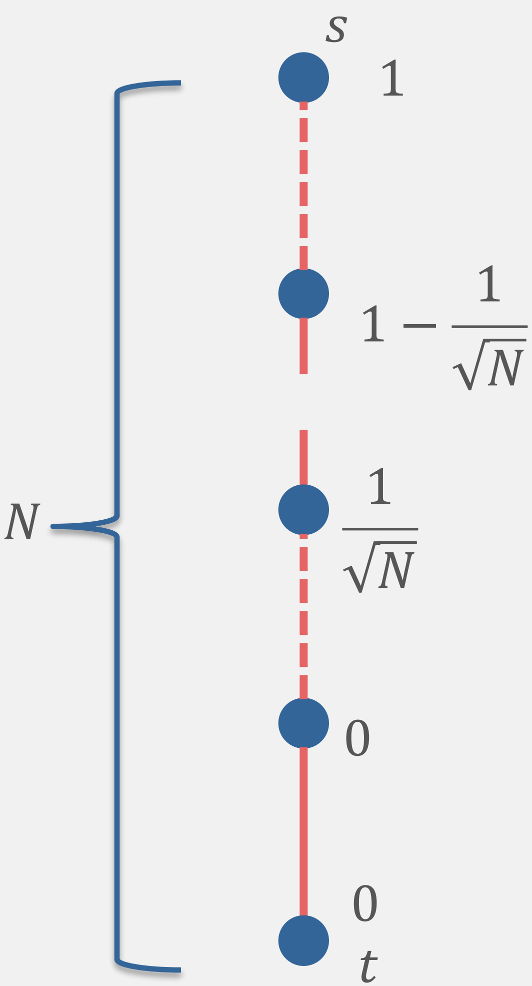

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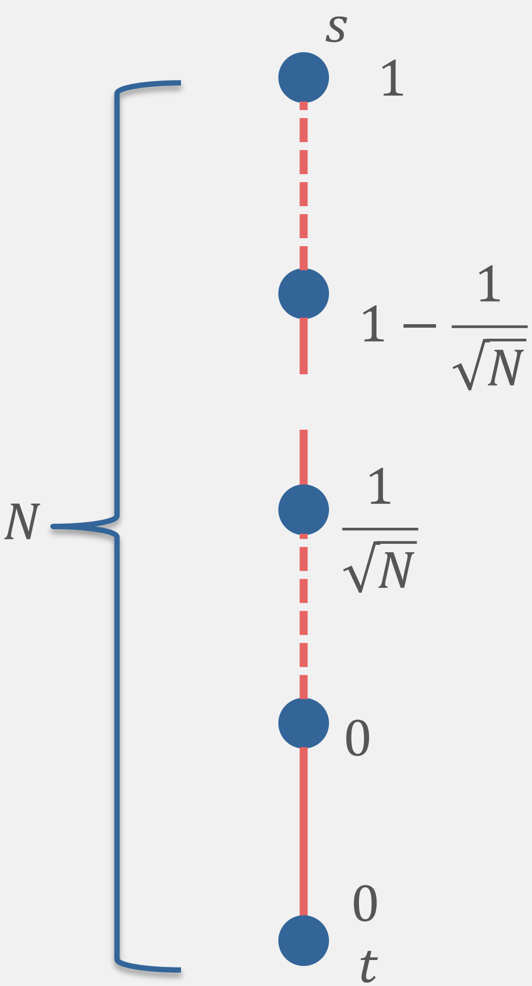

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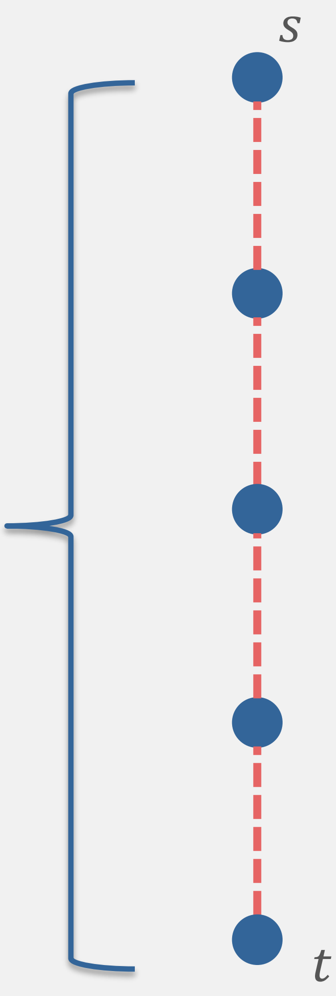
$$\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)}\ \sqrt{\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G')}$$

$$\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G') = \sqrt{N} \times \left(\frac{1}{\sqrt{N}}\right)^2 = \frac{1}{\sqrt{N}}$$

Example



What is quantum complexity of deciding if

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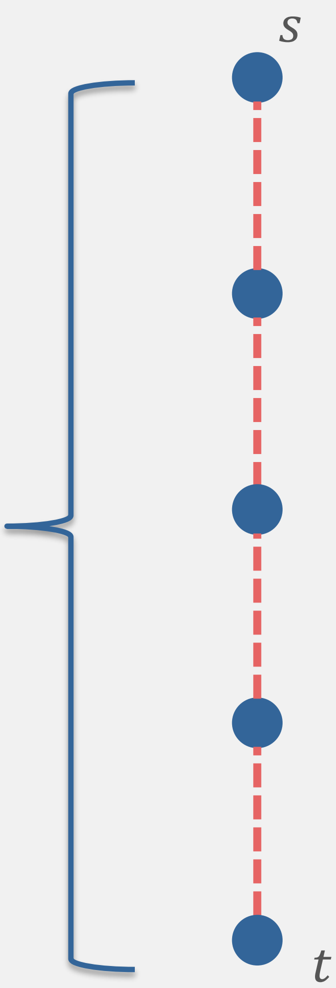
$$\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G)} \qquad \sqrt{\max_{G\in\mathcal{H}:not\ connected} R_{s',t'}(G')}$$

$$N \qquad 1/\sqrt{N}$$

Quantum complexity is $O\left(N^{1/4}\right)$

Example



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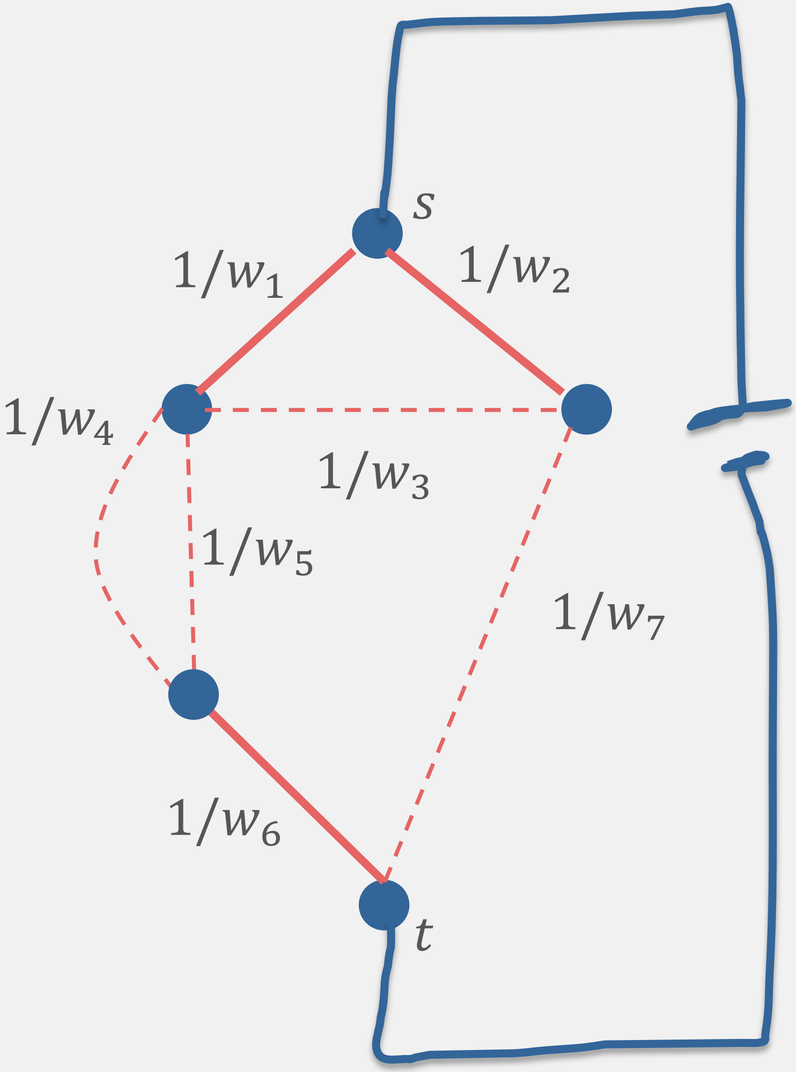
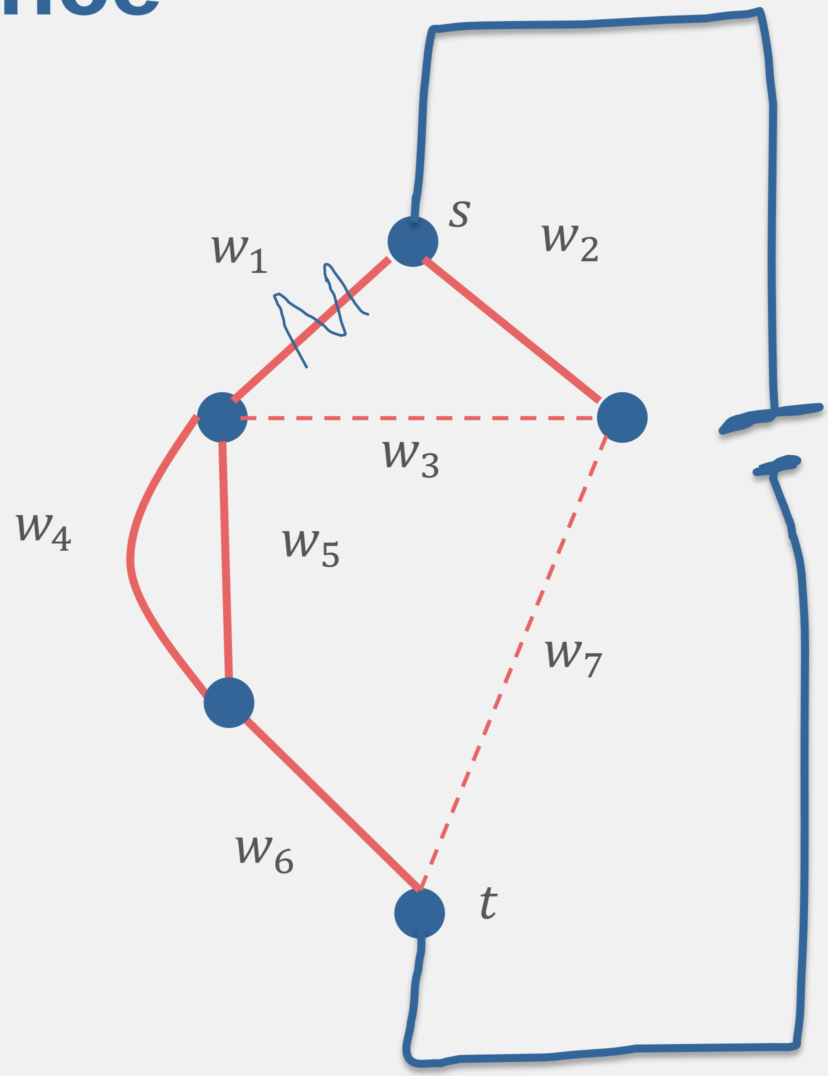
Quantum complexity is $O\left(N^{1/4}\right)$

Randomized classical complexity is $\Omega\left(N^{1/2}\right)$

Algorithm Performance





Algorithm Performance:

st-connectivity algorithm complexity =

$$O\left(\min_{w}\sqrt{\max_{G\in\mathcal{H}:connected} R_{s,t}(G,w)}\sqrt{\max_{G'\in\mathcal{H}:not\ connected} C_{s,t}(G',w)}\right)$$

[JJ**K**P, in progress]

Performance

- Vs. previous quantum st −connectivity algorithm
  - Find a family of graphs with N edges where our analysis uses $O(1)$ queries, previous analysis uses $O\left(N^{1/4}\right)$ queries. [J**K**]
  - Series-parallel graphs, our analysis uses $O\left(N^{1/2}\right)$ queries, previous analysis uses $O(N)$ queries. [J**K**]

- Vs. previous quantum Boolean formula algorithm
  - Match celebrated result that $O\left(\sqrt{N}\right)$ queries required for total read-once Boolean formulas, but proof is simple!
  - Extend super-polynomial quantum to classical speed-up for families of NAND-trees [Z**K**H'12, **K**'13]

Related Algorithms

- Algorithms to estimate capacitance and effective resistance [JJ**K**P]

- Algorithm to decide if graph with n vertices is completely connected, using

$$O\left(\sqrt{Dmn}\right)$$

queries, where promised if connected, has diameter D, or if not connected, largest connected component has M vertices. [JJ**K**P]

Open Questions and Current Directions

- When is our algorithm optimal for Boolean formulas? (Especially partial/read-many formulas)
- Are there other problems that reduce to st-connectivity? (Perhaps all?)
- What is the classical time/query complexity of st-connectivity in the black box model? Under the promise of small capacitance/resistance?
- Does our reduction from formulas to connectivity give good classical algorithms too?
- How to choose weights?

Other interests

- Statistical inference and machine learning applied to quantum characterization problems
- Quantum complexity theory, especially quantum versions of NP